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MATHEMATICS

On Krein Graphs without Triangles

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We consider undirected graphs without loops and multiple edges. If a is a vertex of a graph Γ, then $\Gamma_i(a)$ denotes the subgraph of Γ induced by the set of vertices of Γ that are a distance of i away from a. The subgraph $\Gamma_1(a)$ is called the neighborhood of a and is denoted by $[a]$. Let $a^\perp$ stand for $[a] \cup \{a\}$. The degree of a in Γ is denoted by $k_a = |[a]|$.

An incidence system $(X, \mathcal{B})$ with the set of points X and the set of blocks $\mathcal{B}$ is called a t-(V, K, Λ) scheme if $|X| = V$, each block is incident with exactly K points, and any t points are incident with exactly Λ blocks.

Γ is called a strongly regular graph with parameters (v, k, λ, μ) if Γ is a regular graph of degree k on v vertices; each edge of Γ lies in exactly λ triangles; and, for any two nonadjacent vertices a and b, the subgraph $[a] \cap [b]$ contains exactly μ vertices. For a subgraph Δ of Γ, let $X_i(\Delta)$ denote the set of all vertices of $\Gamma - \Delta$ that are adjacent to exactly i vertices of Δ. For a vertex $y \in \Gamma$, let $\Delta(y)$ denote the subgraph $[[y] \cap \Delta]$. A complete bipartite graph, with its vertex sets having m_1 and m_2 vertices, respectively, is designated as K_{m_1, m_2}. The graph $K_{1, m}$ is called an m-claw.

Suppose that a strongly regular graph Γ with parameters (v, k, λ, μ) has the eigenvalues k, r, and s. If Γ and $\bar{\Gamma}$ are connected graphs, the following inequalities hold, which are known as the Krein conditions:

1. $(r+1)(k+r+2rs) \le (k+r)(s+1)^2$ and
2. $(s+1)(k+s+2rs) \le (k+s)(r+1)^2$.

Γ is called a Krein graph if one of the Krein conditions ((1) or (2)) holds as an equality for it. By Theorem 8.15 in [1], for any vertex a of a Krein graph Γ, the subgraphs $[a]$ and $\Gamma_2(a)$ are strongly regular. Let Γ be a Krein graph without triangles. By Proposition 8.12 and Theorems 8.7 and 8.8 in [1], a strongly regular graph Γ without triangles and with $2 < \mu < k$ has the parameters $((r^2+3r)^2, r^3+3r^2+r, 0, r^2+r)$ if and only if any 3-coclique from Γ belongs to a neighborhood of exactly r vertices. Such a graph is denoted by Kre(r). For any adjacent vertices $a, b \in \Gamma$, the subgraphs $\Gamma_2(a)$ and $\Gamma_2(a) \cap \Gamma_2(b)$ are strongly regular and have the parameters $((r^2+2r-1)(r^3+3r+1), r^3+2r^2, 0, r^2)$ and $((r^2+2r)(r^2+2r-1), r^3+r^2-r, 0, r^2-r)$, respectively. It is well known that, for $r = 1$ and 2, there are unique graphs Kre(r), namely, the Clebsch graph and the Higman–Sims graph, respectively. The latter was constructed in 1968 together with the sporadic simple Higman–Sims group.

Consider a scheme $\mathcal{D} = ([a], \Gamma_2(a))$, where a point $b \in [a]$ is incident with a block $B \in \Gamma_2(a)$ if and only if b and B are adjacent in Γ. Then the 3-scheme $\mathcal{D}$ is an extension of the symmetric 2-scheme $\mathcal{D}' = ([a] - \{b\}, [b] - \{a\})$. If $r = 3$, then $\mathcal{D}'$ is a 2-(56, 11, 2) scheme. Studying a self-orthogonal code over F_3 corresponding to the rows of the incidence matrix of a hypothetical 3-(57, 12, 2) scheme, Bagchi proved that such 3-schemes do not exist (see [2]).

Theorem 1 in [3] states that Kre(r) does not contain any subgraphs $K_{r,r}$; in particular, the graph Kre(3) does not exist. However, Lemma 3 in [3] involves an arithmetical error. We show that Kre(r) does not contain any subgraphs $K_{r,r}$ for $r \ge 9$ (see Lemma 2) and, by using a graph-theoretic argument (possibly applicable to larger r), we prove the nonexistence of Kre(3), i.e., strongly regular graphs with the parameters (324, 57, 0, 12).

Theorem. *There are no strongly regular graphs with the parameters* $((r^2+3r)^2, r^3+3r^2+r, 0, r^2+r)$ *for* $r = 3$.

Corollary. *Symmetric* 2-(56, 11, 2) *schemes are not extendable.*

The following auxiliary result is given without proof (see [3]).

Lemma 1. *Let Γ be a strongly regular graph with parameters (v, k, λ, μ); Δ be an induced subgraph on N vertices and M edges with vertex degrees $d_1, d_2, \ldots, d_N$; and $x_i = |X_i(\Delta)|$. Then,*

$$(v-N) - (kN - 2M) + \left(\lambda M + \mu\left(\binom{N}{2} - M\right) - \sum_{i=1}^{N}\binom{d_i}{2}\right) = x_0 + \sum_{i=3}^{N}\binom{i-1}{2}x_i$$

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and it is true that

$$(1)\ v - N = \sum_{i=0}^{N} x_i;$$

$$(2)\ kN - 2M = \sum_{i=1}^{N} i x_i;$$

$$(3)\ \lambda M + \mu\left(\binom{N}{2} - M\right) - \sum_{i=1}^{N}\binom{d_i}{2}\Big) = \sum_{i=2}^{N}\binom{i}{2}x_i.$$

Throughout the rest of this paper, we assume that Γ is a strongly regular graph with the parameters $((r^2 + 3r)^2, r^3 + 3r^2 + r, 0, r^2 + r)$, where $r \ge 3$.

Lemma 2. *Suppose that* Γ *contains a* $K_{r,r}$*-subgraph* Δ *with* $X_i = X_i(\Delta)$ *and* $x_i = |X_i|$. *Then, the following statements hold*:

(1) $x_0 = -r^3 + 9r^2 - 2r$, $x_1 = 8r^3$, *and* $x_2 = r^4 - r^3$; *in particular*, $r < 9$.

(2) *The neighborhood of a vertex from* $X_0 \cup \Delta$ *contains* $r^3 - r^2$ *vertices from* X_2 *and* $4r^2$ *vertices from* X_1. *The neighborhood of a vertex from* X_1 *contains* $\frac{2r^3 - 5r^2 + 3r}{2}$ *vertices from* X_2, $6r^2 - 5r$ *vertices from* X_1, *and* $\frac{-r^2 + 9r - 2}{2}$ *vertices from* X_0. *The neighborhood of a vertex from* X_2 *contains* $r(r-2)^2$ *vertices from* X_2, $8r^2 - 12r$ *vertices from* X_1, *and* $-r^2 + 9r - 2$ *vertices from* X_0.

(3) *If* u *and* w *are two nonadjacent vertices of* X_0 *and* $|[u] \cap X_0(w)| = i$, *then* $|[u] \cap X_1(w)| = 2r - 2i$ *and* $|[u] \cap X_2(w)| = r^2 - r + i$.

(4) *If* r *is odd, then, for any pair of nonadjacent vertices* $u, w \in X_0$, *the subgraph* $[u] \cap [w] \cap X_0$ *contains no more than* $r - 1$ *vertices. If* $r = 3$, *then such a subgraph contains no more than one vertex.*

Proof. Let Δ be a $K_{r,r}$-subgraph of Γ, $X_i = X_i(\Delta)$, and $x_i = |X_i|$. Since each 3-coclique from Δ is in the neighborhood of r vertices of Δ, then $x_i = 0$ for $i \ge 3$. Next, by using Lemma 1, the resulting system of equations is solved for x_i, and, since they are nonnegative, we arrive at statement (1).

We pick a vertex $f \in X_0$ and set $Y_i = X_i - f^{\perp}$ and $y_i = |Y_i|$. Applying Lemma 1 to the subgraph Δ of $\Gamma_2(f)$ yields $y_0 = -r^3 + 9r^2 - 3r - 1$, $y_1 = 8r_3 - 4r^2$, and $y_2 = r^4 - 2r^3 + r^2$. Therefore, $[f]$ contains r vertices of X_0, $4r^2$ vertices of X_1, and $r^3 - r^2$ vertices of X_2.

Let $a \in \Delta$. Then, $|\Delta - a^{\perp}| = r - 1$ and $[a] \cap [b]$ contains r^2 vertices from X_2 for $b \in \Delta - a^{\perp}$. Therefore, $|[a] \cap X_2| = r^3 - r^2$, $|[a] \cap X_1| = 4r^2$, and $|\Delta(a)| = r$.

Let $b \in X_1$. Then, $[b]$ contains a vertex from Δ and $\binom{r}{2}(r-1) + \binom{r-1}{2}r = \frac{2r^3 - 5r^2 + 3r}{2}$ vertices from X_2. Moreover, $[b]$ contains $3r - 2$ vertices from X_1 that are adjacent to a vertex from the vertex subset of Δ that does not include b, and $[b]$ contains $3r$ vertices from X_1 that are adjacent to a vertex from the other vertex subset of Δ. Therefore, $[b]$ contains $6r^2 - 5r$ vertices from X_1 and $\frac{-r^2 + 9r - 2}{2}$ vertices from X_0.

Let $c \in X_2$. Then, $[c]$ contains two vertices from Δ and $\binom{r}{2}(r-2) + \binom{r-2}{2}r = r(r-2)^2$ vertices from X_2. Moreover, $[c]$ contains $4r - 4$ vertices from X_1 that are adjacent to a vertex from the vertex subset of Δ that does not include c, and $[c]$ contains $4r$ vertices from X_1 that are adjacent to a vertex from the other vertex subset of Δ. Therefore, $[c]$ contains $8r^2 - 12r$ vertices from X_1 and $-r^2 + 9r - 2$ vertices from X_0. Statement (2) is proved.

Let u and w be nonadjacent vertices from X_0. Then, for $y \in \Delta$, the subgraph $[y] \cap [u] \cap [w]$ contains r vertices (altogether, $2r^2$ vertices). If $[u] \cap [w]$ contains i vertices from X_0, then $|[u] \cap [w] \cap (X_1 \cup X_2)| = r^2 + r - i$. Therefore, $|[u] \cap [w] \cap X_2| = r^2 - r + i$ and $|[u] \cap [w] \cap X_1| = 2r - 2i$. Statement (3) is proved.

For a vertex $a_i \in \Delta$, the subgraph $[a_i] \cap [u] \cap [w]$ contains r vertices. Let δ_i be the number of vertices in $X_i(\{a_1, a_2, \ldots, a_r\}) \cap [u] \cap [w]$. Then, $\delta_1 + 2\delta_2 = r^2$. If r is odd, we have $\delta_2 \le \frac{r^2 - 1}{2}$. Therefore, $[u] \cap [w]$ contains no less than two vertices from X_1 and no more than $r - 1$ vertices from X_0.

Let $r = 3$, $w \in X_0$, $\{u_1, u_2, u_3\} = [w] \cap X_0$, $\nu = |[u_1] \cap [u_2] \cap [u_3] \cap X_2|$, and $\nu \le 2$. By $X_0^2(w)$, we denote the set of vertices that are a distance of 2 away from w in X_0. Suppose that $X_0^2(w)$ contains exactly s vertices $z_1, z_2, \ldots, z_s$ such that $|[z_i] \cap [w] \cap X_0| = 2$. Then, $|[z_i] \cap [w] \cap X_2| = 8$.

If $s = 3$, we may assume that u_i is nonadjacent to z_i. Since $X_2 \cap [u_1]$ contains $\nu + 2(8 - \nu)$ vertices from $[u_2] \cup [u_3]$, we conclude that $|X_2(u_1) \cap [z_1]| \le 18 - \nu - 2(8 - \nu) \le 4$, which contradicts the fact that $[u_1] \cap [z_1]$ contains no less than six vertices from X_2. If $s = 2$, we may assume that $\{u_2, u_3\} = X_0(w) \cap [z_1]$ and $\{u_1, u_3\} = X_0(w) \cap [z_2]$. Similarly, $|X_2(u_1) \cap [z_1]| \le 3 + \nu \le 5$, which is a contradiction.

If $s = 1$, we may assume that $u_1, u_2 \in [z_1]$. Then, the number of vertices in $X_2 \cap (\bigcup_i [u_i] \cup [w])$ is $50 + \nu$. Furthermore, $|[u_3] \cap [z_1] \cap X_2| \le 4 + \nu \ge 6$ and $\nu = 2$. Thus, $X_2 \cap [z_1]$ contains six vertices from $[u_3]$, eight vertices from $[w]$, and another no more than $54 - (50 + \nu) = 2$ vertices. This contradicts the fact that each vertex in X_0 is adjacent to 18 vertices of X_2. The lemma is proved.

Suppose that $r = 3$, Γ contains a $K_{3,3}$-subgraph $\Delta = \{a_1, a_2, a_3; b_1, b_2, b_3\}$, $X_i = X_i(\Delta)$, $A^{ij} = [a_i] \cap [a_j] \cap X_2$, $B^{lm} = [b_l] \cap [b_m] \cap X_2$, and $w \in X_0$.

Lemma 3. *For vertices c, $d \in X_2$, the following is true*:

(1) *If $[d] \cap [w]$ contains i vertices from X_0, then* $|[d] \cap [w] \cap X_2| = i$.

(2) *One of the following statements holds*:

(i) $|[c] \cap [d] \cap \Delta| = 2$ *and* $|[c] \cap [d] \cap X_0| = 4$;

(ii) $|[c] \cap [d] \cap \Delta| = 0$ *and* $|[c] \cap [d] \cap X_0| = 6$;

(iii) $|[c] \cap [d] \cap \Delta| = 1$ *and*, if $[c] \cap [d]$ *contains t vertices from X_2, then* $|[c] \cap [d] \cap X_0| = 5 + t$; *moreover, if $t \geq 2$, then c and d are in the $K_{3,3}$-subgraph X_2 of Δ' and $X_2(\Delta') = X_0 \cup \Delta$.*

Proof. Suppose that $|[d] \cap [w] \cap X_0| = i$ and, for definiteness, let $d \in A^{12}$. Then, $[d] \cap [w]$ contains three vertices from $[a_3]$ and three vertices from $[b_i]$. Therefore, $|[d] \cap [w] \cap X_2| = i$ and $|[d] \cap [w] \cap X_1| = 12 - 2i$. Statement (1) is proved.

Let $[c] \cap [d]$ contain a_1 and a_2. Then, $[c] \cap [d]$ contains one vertex from each $X_1 \cap [b_i]$, three vertices from $[a_3]$, and four vertices from X_0. Let $c \in A^{12}$ and $d \in B^{12}$. Then, $[c] \cap [d]$ contains three vertices from $[a_3]$, three vertices from $[b_3]$, and six vertices from X_0.

Let $c \in A^{12}$ and $d \in A^{13}$. Then, $[c] \cap [d]$ contains two vertices from each $[b_i]$. If $[c] \cap [d]$ contains t vertices from X_2, then $[c] \cap [d]$ contains $5 + t$ vertices from X_0. Assume that $t \geq 2$ and $[c] \cap [d] \cap X_2$ contains two vertices f and g. If $|[c] \cap [d] \cap X_2| = 2$, then $|[c] \cap [d] \cap X_2| = 7$ and $[c] \cup [d]$ contains $7 + 2 \cdot 9$ vertices from X_0. Symmetrically, $[f] \cup [g]$ contains no less than $8 + 2 \cdot 8$ vertices from X_0. This is a contradiction with $|X_0| = 48$. Therefore, $t = 3$ and the vertices c and d are in the $K_{3,3}$-subgraph Δ' of X_2. Since $48 = 6 \cdot 8$, each vertex in X_0 is adjacent to exactly two vertices from Δ' and $X_2(\Delta') = X_0 \cup \Delta$. The lemma is proved.

Fix a vertex $d \in A^{12}$. Suppose that $\{e_1, e_2, e_3\} = X_2(d)$ and $[e_i] \cap [e_j]$ contains a single vertex from X_2. Let X_0^{ij} be the set of vertices from X_0 that are adjacent only to the vertices e_i and e_j from $\{e_1, e_2, e_3\}$; X_0^i be the set of vertices from X_0 that are adjacent only to the vertex e_i; $X_0^0 = X_0 - (\bigcup_i [e_i] \cup [d])$; and $\{d, f_{i1}, f_{i2}\} = X_2(e_i)$, where $f_{1j}, \ldots, f_{3j} \in A^{j3}$.

Lemma 4. *Let* $\Psi = \Delta \bigcup_i \{d, e_1, e_2, e_3\}$, $Y_i = X_i(\Psi)$, *and* $y_i = |Y_i|$.

Then, $y_0 = y_4 = 0$, $|X_0^{ij}| = 4$, $|X_0^i| = 6$, *and* $|X_0^0| = 0$.

Proof. Let g be a vertex from $\Gamma - \Psi$. If $g \in Y_5$, then $[g]$ contains e_1, e_2, e_3, and two vertices from $\{a_1, a_2, a_3\}$, which is a contradiction. If $g \in Y_4$, then $[g]$ contains e_1, e_2, e_3, and a vertex a_i (where $i = 1$ or 2) or a_3. In the former case, for definiteness, let $i = 1$. Then, we obtain a $K_{3,3}$-subgraph $\Phi = \{a_1, e_1, e_2; b_1, d, g\}$; moreover, b_2, $b_3 \in X_2(\{a_1, e_1, e_2\})$, $a_2, e_3 \in X_2(\{b_1, d, g\})$, and $[b_2] \cap [b_3]$ contains a_2 and e_3. By Lemma 3, the subgraph $\Phi \cup \{a_2, e_3, b_2, b_3\}$ is contained in a $K_{6,6}$-subgraph with the maximal matching removed. This is a contradiction with the fact that a_3 is the antipode of d in the $K_{6,6}$-subgraph and a_1 is adjacent to g.

Therefore, g is adjacent to a_3. If $|\bigcap_i [e_i] \cap X_0| = s$, then X_0 contains $6 - s$ vertices from $[e_2] \cap [e_3]$, $4 + s$ vertices from $[e_2]$–$[e_3]$, and $4 + s$ vertices $[e_3]$–$[e_2]$. It follows that $y_0 = 2 - s$. Since $[e_1] \cap [e_2] \cap [e_3]$ contains two vertices from $Y_4 \cup X_0$, $y_4 = y_0$. Furthermore, $[d] \cap [g]$ does not intersect $[a_i]$ for $i = 1$ and 2 and $[d] \cap [g] \cap [b_j]$ contains two vertices from $\{e_1, e_2, e_3\}$ and a vertex from X_1. Therefore, $|[d] \cap [g] \cap X_0| = 6$ and $|[g] \cap Y_0| = 2$.

If $y_0 = 0$, then, by Lemma 3, we obtain $|[e_i] \cap [e_j] \cap X_0| = 6$. Therefore, $|X_0^{ij}| = 4$.

Consider the case $y_0 = 2$. Define $\{g_1, g_2\} = Y_4$ and $\{o_1, o_2\} = [g_1] \cap [g_2] \cap X_0 = Y_0$. Let $[f_{11}]$ contain exactly t vertices from X_0^{23}. By Lemma 3, for $t = 2$, $[f_{11}]$ contains four vertices from X_0^2, four vertices from X_0^3, and does not intersect X_0^0. This is a contradiction with the fact that $[d] \cap [f_{11}] \cap [g_1]$ contains e_1 and no less than three vertices from X_0. Therefore, $t = 3$ and $[f_{11}]$ contains three vertices from X_0^2, three vertices from X_0^3, and a single vertex from X_0^0. Moreover, $X_0(d) \cap [f_{ij}]$ contains two vertices in each of $[g_1]$ and $[g_2]$.

Assume that $[o_1] \cap [o_2]$ contains a vertex from X_0. Then, $[o_1] \cap [o_2]$ contains seven vertices from X_2 and four vertices from X_1 (including g_1 and g_2). The subgraph $[o_1] \cap [o_2] \cap [e_i]$ does not intersect X_2 and contains g_1, g_2, and one more vertex from $X_0 \cup X_1$. If a vertex other than g_1 or g_2 is adjacent to o_1, o_2, e_1, and e_2, then we obtain a $K_{3,4}$-subgraph. Therefore, each vertex in $[o_1] \cap [o_2] \cap (X_0 \cup X_1 - \{g_1, g_2\})$ is adjacent to a single vertex of $\{e_1, e_2, e_3\}$, which contradicts the fact that $[o_1] \cap [o_2] \cap [d]$ contains three vertices from $X_0 \cup X_1$.

Therefore, $|[o_1] \cap [o_2] \cap X_0| = 0$. Then, $[o_1] \cap [o_2]$ contains six vertices from each of X_1 and X_2. Specifically, $[o_1] \cap [o_2]$ contains g_1, g_2, three vertices from $X_1(d)$, and the vertex c from X_1. This contradicts the fact that $[o_1] \cap [o_2] \cap [e_i] = \{g_1, g_2, c\}$ for any $i \in \{1, 2, 3\}$.

Lemma 5. *The subgraph X_2 contains a single $K_{3,3}$-subgraph Δ'. Moreover, if $\{w; u_1, u_2, u_3\}$ is a 3-claw in X_0 and $c \in X_2$ is an arbitrary vertex nonadjacent to any of the vertices of this 3-claw, then* $(\{c\} \cup X_2(c)) \subset \Delta'$.

Proof. Suppose that X_2 contains a $K_{3,3}$-subgraph Δ'. By Lemma 3, $X_2(\Delta') = X_0 \cup \Delta$; therefore, X_2 contains a single $K_{3,3}$-subgraph.

Let c in X_2 not be adjacent to any of the vertices of the 3-claw $\{w; u_1, u_2, u_3\}$ from X_0. By statement (1) in Lemma 3, w is nonadjacent to the vertices of $\{c\} \cup X_2(c)$. Let $\{g_1, g_2, g_3\} = X_2(c)$ and $\Psi' = \Delta \cup \{c\} \cup X_2(c)$. If $|[g_1] \cap [g_2] \cap X_2| = 1$, then Lemma 4 implies that the subgraph $\bigcap_i [g_i] \cup [c]$ contains X_0, which contradicts the fact that w does not belong to it. Now, by Lemma 3, the subgraphs $\{c\} \cup X_2(c)$ is contained in the $K_{3,3}$-subgraph Δ' from X_2. The lemma is proved.

The proof of the theorem is completed as follows. Let $a \in \Gamma$ and $u \in \Gamma_2(a)$. By Lemma 5, any three vertices c_1, c_2, and c_3 in $[a] \cap [u]$ are contained in the single $K_{6,6}$-subgraph $\Omega = \Omega(c_1, c_2, c_3)$ with the maximal matching removed (of course, it contains a and u). Then, $\Omega(a) \cap \Omega(u)$ contains the fourth vertex c_4, and this quadruple of vertices can be uniquely recovered from any of its triples. Let $X = [a] \cap [u]$ and $\mathcal{B}$ be the set of indicated quadruples from X. Then, $(X, \mathcal{B})$ is a 3-(12, 4, 1) scheme with $b = |\mathcal{B}| = \binom{12}{3} \Big/ \binom{4}{3} = 55$, which is a contradiction with the fact that each point in X lies in exactly $\frac{4b}{12} = \frac{55}{3}$ blocks. The theorem is proved.

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