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Distance-regular graphs with intersection arrays {52, 35, 16; 1, 4, 28} and {69, 48, 24; 1, 4, 46} do not exist

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Abstract We prove that the arrays $\{52, 35, 16; 1, 4, 28\}$ and $\{69, 48, 24; 1, 4, 46\}$ cannot be realized as the intersection arrays of distance-regular graphs. In the proof we use some inequalities bounding the size of substructures (cliques, cocliques) in a distance-regular graph.

Keywords Distance-regular graph · Intersection array · Terwilliger graph

Mathematics Subject Classification 05E30

1 Introduction

There are many necessary conditions (combinatorial and algebraic) that sometimes allow to decide whether a distance-regular graph with a given intersection array does exist. Some of them concern the restrictions of extremal subgraphs (cliques and cocliques) embeddings. In this note we involve such conditions to establish the following result.

Theorem 1 *There exists no distance-regular graph with intersection array* $\{52, 35, 16; 1, 4, 28\}$.

Theorem 2 *There exists no distance-regular graph with intersection array* $\{69, 48, 24; 1, 4, 46\}$.

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The first array is from the list of feasible intersection arrays, see [1, Chap. 14]. The second one is also feasible and appears in [2, Theorem 19], where the so-called Shilla distance-regular graphs with $b(\Gamma) = 3$ are classified.

Our work is motivated by the paper [2], in which a new existence condition for distance-regular graphs was proposed (see Lemma 2 below). In [3,4], applying this condition and combinatorial arguments, we ruled out some feasible intersection arrays from the list of [1, Chap. 14].

The paper is organized as follows. In Sect. 2, we give the definitions and recall two important inequalities bounding the size of cliques and cocliques in a distance-regular graph. In Sect. 3, we apply these bounds (together with some combinatorial arguments) to prove our theorems.

2 Preliminaries

We consider only finite undirected graphs without loops or multiple edges. Let Γ be such a graph. The *distance* $\mathrm{d}(x, y)$ between any two vertices x and y of Γ is the length of a shortest path from x to y in Γ. For vertices x, y, we write $x \sim y$, if x and y are adjacent (i.e., $\mathrm{d}(x, y) = 1$), and $x \not\sim y$, if they are not. The *diameter* $\mathrm{diam}(\Gamma)$ of Γ is the maximal distance occurring in Γ.

For a subset X of the vertex set of Γ, we will also write X for the subgraph of Γ induced by X. For a vertex x of Γ, define $\Gamma_i(x)$ to be the set of vertices that are at distance i from x $(0 \le i \le \mathrm{diam}(\Gamma))$. The *valency* of x is the number of neighbors of x, i.e., $|\Gamma_1(x)|$. A graph is *regular* with valency k if the valency of each of its vertices is k. We will write $\Gamma(x)$ instead of $\Gamma_1(x)$ for short. For vertices $x_1, x_2, \ldots, x_n \in \Gamma$, define $\Gamma(x_1, x_2, \ldots, x_n) := \cap_{i=1}^{n} \Gamma(x_i)$.

A connected graph Γ with diameter $d := \mathrm{diam}(\Gamma)$ is *distance-regular* if there are integers b_i, c_i $(i = 0, \ldots, d)$ such that, for every pair of vertices $x, y \in \Gamma$ with $\mathrm{d}(x, y) = i$, there are exactly c_i neighbors of x in $\Gamma_{i-1}(y)$ and b_i neighbors of x in $\Gamma_{i+1}(y)$ (we assume that $\Gamma_{-1}(y)$ and $\Gamma_{d+1}(y)$ are empty sets). In particular, a distance-regular graph Γ is regular with valency b_0, and $c_1 = 1$. For each vertex $x \in \Gamma$ and $i = 0, \ldots, d$, the subgraph $\Gamma_i(x)$ is regular with valency $a_i := b_0 - b_i - c_i$. The numbers a_i, b_i, c_i $(i = 0, \ldots, d)$ are called the *intersection numbers* and the array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ is called the *intersection array* of the distance-regular graph Γ.

For a graph Γ, the eigenvalues of its adjacency matrix are called the *eigenvalues* of Γ. We recall that a distance-regular graph with diameter d has $d + 1$ distinct eigenvalues exactly, which can be calculated from its intersection array [1, Sect. 4.1.B].

An *l-clique* L of Γ is a complete subgraph (i.e., every two vertices of L are adjacent) of Γ with exactly l vertices. We say that L is a clique if it is an l-clique for certain l. A coclique C of Γ is an induced subgraph of Γ with empty edge set.We say a coclique is a *c-coclique* if it has exactly c vertices.

Denote by $(n \times m)$-*grid* the graph with as vertices the pairs (i, j), $1 \le i \le n$, $1 \le j \le m$, where two distinct pairs (i, j) and (i', j') are adjacent if and only if $i = i'$ or $j = j'$.

A *Terwilliger graph* is a connected non-complete graph Γ that does not contain an induced quadrangle. There are only three distance-regular Terwilliger graphs known with $c_2 \ge 2$: the icosahedron with intersection array $\{5, 2, 1; 1, 2, 5\}$, the Doro graph with intersection array $\{10, 6, 4; 1, 2, 5\}$, the Conway–Smith graph with intersection array $\{10, 6, 4, 1; 1, 2, 6, 10\}$ (all of them are characterized by their intersection arrays).

Let us recall two inequalities that bound the size of cliques or cocliques in a distance-regular graph. Let Γ be a distance-regular graph with diameter $d \ge 2$ and smallest eigenvalue

θ. The following result is due to several authors (Hoffman [1, Proposition 4.4.6(i)] and Delsarte [5]).

Lemma 1 *The size of a clique* L *in* Γ *is bounded by*

$$|L| \leq 1 - b_0/\theta,$$

and in case of equality every vertex $x \notin L$ *is adjacent to either* 0 *or* $1 - b_0/\theta + b_1/(\theta+1)$ *vertices of* L.

Lemma 2 *If, for a vertex* $x \in \Gamma$, *the subgraph* $\Gamma(x)$ *contains a* c*-coclique,* $c \geq 2$, *then*

$$c_2 - 1 \geq \frac{c(a_1+1) - b_0}{\binom{c}{2}}.$$

Remark 1 Lemma 2 was also proved by several authors, see [2,6]. In addition, in [2, Proposition 3] it was shown that if c is maximal such that for all $x \in \Gamma$ and all $y, z \in \Gamma(x)$ with $y \not\sim z$, there exists a coclique of size at least c in $\Gamma(x)$ containing y and z, then

$$c_2 - 1 \geq \max\left\{\frac{c'(a_1+1) - b_0}{\binom{c'}{2}} \mid 2 \leq c' \leq c\right\},$$

and equality implies that Γ is a Terwilliger graph. The case of equality was considered in [7] (in fact, if $c_2 \geq 2$ then Γ is the icosahedron, the Doro graph or the Conway–Smith graph).

3 Main results

From now on, we suppose that Γ is a counterexample to Theorem 1. Then Γ has spectrum $52^1, 20^{65}, 4^{182}, -4^{520}$.

Choose the vertices $x, y \in \Gamma$ at distance 2, and put $X := \Gamma(x) \cap \Gamma_2(y)$, $Y := \Gamma(y) \cap \Gamma_2(x)$. Then $|X| = |Y| = a_2 = 32$.

Lemma 3 *The following holds.*

1. *a clique of* Γ *contains at most 13 vertices;*
2. $\Gamma(x)$ *does not contain a 5-coclique;*
3. $\Gamma(x, y)$ *does not contain a 3-coclique;*
4. *if* $z \in \Gamma(x, y)$ *then* $|\Gamma(x, y, z)| \geq 2$.

Proof Lemmas 1 and 2 give Statements (1) and (2), respectively. If $\Gamma(x, y)$ contains a 3-coclique then, by (2), the subgraph $\Gamma(x) \cap \Gamma_3(y)$ is a 16-clique, which is impossible by (1). This gives (3).

Let us prove (4). Suppose z is a vertex of $\Gamma(x, y)$ such that $|\Gamma(x, y, z)| \leq 1$; then $|X \backslash \Gamma(z)| = |X| - (a_1 - |\Gamma(x, y, z)|) = 16 + |\Gamma(x, y, z)| \leq 17$ (similarly, $|Y \backslash \Gamma(z)| \leq 17$). Consider two vertices u_1 and u_2 of $\Gamma(x, y) \backslash \Gamma(z)$. Since $\Gamma(u_i)$ contains at least 27 vertices of $(X \cup Y) \backslash \Gamma(z)$,we see that $\Gamma(u_1, u_2)$ contains x, y and at least $20 > \max\{a_1, c_2\}$ vertices of $(X \cup Y) \backslash \Gamma(z)$, a contradiction. The lemma is proved. □

Lemma 4 *Suppose* z_1 *and* z_2 *are two non-adjacent vertices of* $\Gamma(x, y)$*; then* $\Delta := \Gamma(x) \backslash (\Gamma(z_1) \cup \Gamma(z_2) \cup \{z_1, z_2\})$ *is a* (2×10)*-grid.*

Proof It follows from Lemma 3(2) that Δ does not contain a 3-coclique. Now, by Lemma 3(4), $|\Gamma(x, y, z_i)| \geq 2$ holds, and we see that $|\Gamma(x, y, z_1, z_2)| = 2$. Therefore, the subgraph Δ contains exactly $52 - 2 \cdot (16 - 2) - 4 = 20$ vertices.

Let p_1, p_2 be a pair of non-adjacent vertices of Δ. Then $|\Delta| = 2 + |\Delta(p_1)| + |\Delta(p_2)| - |\Delta(p_1, p_2)|$. Since $|\Gamma(z_i, p_j, x)| \in \{2, 3\}$ holds for all i, j, we have $|\Delta(p_j)| \in \{10, 11\}$ and $|\Delta(p_1, p_2)| \in \{2, 3\}$ (so that Δ is connected). In particular, if $|\Delta(p_1, p_2)| = 2$, then $|\Delta(p_1)| = |\Delta(p_2)| = 10$ and $\Gamma(z_i, p_j) \subset \Gamma(x) \cup \{x\}$. If $|\Delta(p_1, p_2)| = 3$, then, without loss of generality, we may assume that $|\Delta(p_1)| = 11$ and $|\Delta(p_2)| = 10$.

Choose one more vertex p_3 of $\Delta_2(p_1)$ (that is, $p_2 \sim p_3$).

Claim With the notation $\delta := |\Delta(p_1, p_2, p_3)|$, the subgraph $\Delta(p_1)\backslash\Delta(p_2, p_3)$ is a $(|\Delta(p_1)| - \delta)$-clique.

If, without loss of generality, $\Delta(p_1, p_2) \subseteq \Delta(p_1, p_3)$, then $(\Delta(p_1)\backslash\Delta(p_2, p_3)) \subset \Delta_2(p_2)$, and the claim follows, since Δ does not contain a 3-coclique. For every pair of vertices $q \in \Delta(p_1, p_2)\backslash\Delta(p_3)$ and $q' \in \Delta(p_1, p_3)\backslash\Delta(p_2)$, we have $(\Delta(p_1)\backslash(\Delta(p_2) \cup \Delta(p_3))) \subset \Delta(q, q')$. Since $|\Delta(p_1)\backslash(\Delta(p_2) \cup \Delta(p_3))| \geq 11 - 2(3 - \delta) - \delta = 5 + \delta > c_2$, it follows that $q \sim q'$. The same conclusion can be drawn for a pair of vertices $q, q' \in \Delta(p_1, p_i)\backslash\Delta(p_j)$, $\{i, j\} = \{2, 3\}$. Finally, note that $\Delta(p_1)\backslash(\Delta(p_2) \cup \Delta(p_3))$ is a clique. The claim is proved.

Claim $|\Delta(p_1, p_2)| \neq 3$.

Assuming the converse, first we shall show that $\delta \geq 2$. If $\delta = 0$, then it follows from the previous claim that $\Delta(p_1)$ is an 11-clique, a contradiction. If $\delta = 1$, then, for a vertex $q \in \Delta(p_1, p_2)\backslash\Delta(p_3)$, $\Gamma(p_3, q)$ contains x, p_2 and the two vertices of $\Delta(p_1, p_3)\backslash\Delta(p_2)$. We now have $|\Gamma(p_3, q, p_2)| = 1$, contrary to Lemma 3(4).

Thus, $\delta \geq 2$ holds and there are at least 14 edges between the sets $\Delta(p_1, p_2)$ and $\Delta_2(p_1)\backslash\{p_2\}$. We note that if a vertex $r \in \Delta(p_1, p_2)$ has at least 3 neighbors in $\Delta_2(p_1)\backslash\{p_2\}$, then $\Delta_2(p_1) \subset \Delta(r)$ (recall that $\Delta_2(p_1)$ is a clique). Since $14 > 7 + 2 + 2$, it follows that there are at least two vertices $r_1, r_2 \in \Delta(p_1, p_2)$ such that $\Delta_2(p_1) \subset \Delta(r_i)$, $i = 1, 2$. Further, for every vertex $p \in \Delta_2(p_1)$, there is a vertex r from $\Delta(p_1, p)\backslash\{r_1, r_2\}$, and, by Lemma 3(4), we have $r \sim r_i$ for some $i \in \{1, 2\}$. At the same time, the subgraph $\Delta(r_i)$ contains p_1, the eight vertices of $\Delta_2(p_1)$, the vertex r_j, $\{i, j\} = \{1, 2\}$, and, hence, at most one more vertex. Therefore, there is a vertex $r_3 \in \Delta(p_1)\backslash\{r_1, r_2\}$ such that $\Delta_2(p_1) \subset \Delta(r_3)$, and the vertices r_1, r_2, r_3 induce a clique R isolated in $\Delta(p_1)$. Further, for any vertex $r \in \Delta(p_1)\backslash R$, the subgraph $\Delta(r)$ contains p_1 and at most 7 vertices of $\Delta(p_1)\backslash R$, which contradicts $|\Delta(r)| \geq 10$. The claim is proved.

Claim Δ is a (2×10)-grid.

We now see that $|\Delta(p_1, p_2)| = 2$ and the graph Δ is regular with valency 10. It remains to prove that $\Delta(p_1, p_2)$ is a 2-coclique.

As before, if $\delta = 0$, then $\Delta(p_1)$ is a 10-clique and hence Δ is disconnected, a contradiction. Hence, $\delta \geq 1$ holds and there are at least 8 edges between the sets $\Delta(p_1, p_2)$ and $\Delta_2(p_1)\backslash\{p_2\}$. Therefore, there is a vertex $r \in \Delta(p_1, p_2)$ such that $\Delta_2(p_1) \subset \Delta(r)$. This completes the proof. □

Lemma 5 *If the subgraph* $\Gamma(x, y)$ *contains a pair of non-adjacent vertices, say,* z_1 *and* z_2, *then* Γ *contains a clique of size* 14.

Proof By Lemma 4, the subgraph $\Delta := \Gamma(x)\backslash(\Gamma(z_1) \cup \Gamma(z_2) \cup \{z_1, z_2\})$ is a (2×10)-grid. Denote by L_1, L_2 the two distinct 10-cliques of Δ.

Claim If, for a vertex $q \in \Gamma(z_1) \cap X$, $|\Gamma(q) \cap \Delta| \geq 5$ holds and $L_i \not\subset \Gamma(q)$ for all i, then $\Gamma(q) \cap \Delta$ is a (2×3)-grid.

Note that the inequality $|\Gamma(q) \cap L_i| > 3$ implies that $L_i \subset \Gamma(q)$. Therefore, we may assume $|\Gamma(q) \cap L_1| = 3$ and $|\Gamma(q) \cap L_2| \in \{2, 3\}$. If there is a pair of vertices $p \in \Gamma(q) \cap L_1$, $p' \in \Delta(p) \cap L_2$ such that $p' \not\sim q$, then $\Gamma(q, p') = (\Gamma(q) \cap L_2) \cup \{x, p\}$ and, hence, $|\Gamma(q, p', p)| = 1$, contrary to Lemma 3(4). The claim is proved.

Claim If, for a vertex $q \in \Gamma(z_1) \cap X$, $|\Gamma(q) \cap \Delta| = 4$ holds, then $\Gamma(q) \cap \Delta$ is a (2×2)-grid.

Suppose that $\Gamma(q) \cap \Delta$ is not a (2×2)-grid. We will show that there is a vertex $p \in \Delta \backslash \Gamma(q)$ (i.e., $p \not\sim q$) such that either $|\Gamma(q, p)| > c_2 = 4$ or the subgraph $\Gamma(q, p)$ contains a vertex r such that $|\Gamma(p, q, r)| < 2$ (contrary to Lemma 3(4)). Let $L_1 := \{s_1, s_2, \ldots, s_{10}\}$, $L_2 := \{t_1, t_2, \ldots, t_{10}\}$, and $s_i \sim t_i, i = 1, \ldots, 10$.Without loss of generality, for the subgraph $\Gamma(q) \cap \Delta$, one of the following holds:

- $\Gamma(q) \subset L_1$ (for this case, let p be a vertex of $L_1 \backslash \Gamma(q)$),
- $\Gamma(q) \cap \Delta = \{s_1, s_2, s_3, t_4\}$ (take $p = s_4$),
- $\Gamma(q) \cap \Delta = \{s_1, s_2, s_3, t_1\}$ (take $p = t_2$ and $r = s_2$),
- $\Gamma(q) \cap \Delta = \{s_1, s_2, t_3, t_4\}$ (take $p = t_1$ and $r = s_1$),
- $\Gamma(q) \cap \Delta = \{s_1, s_2, t_2, t_3\}$ (take $p = t_1$ and $r = s_1$).

Claim Let q_1, q_2 be a pair of vertices of $\Gamma(z_1) \cap X$ such that, without loss of generality, $\Gamma(q_1) \cap \Delta$ is a (2×3)-grid and $\Gamma(q_2) \cap \Delta$ is a (2×2)-grid or (2×3)-grid. Then $q_1 \not\sim q_2$ and $\Gamma(q_1, q_2) \cap \Delta$ is an empty set.

Note that, for a pair of non-adjacent vertices $p, p' \in \Gamma(q_i) \cap \Delta$, we have $\Gamma(p, p') = \Delta(p, p') \cup \{x, q_i\}$. Therefore, $\Gamma(q_1, q_2) \cap \Delta$ is a 2-clique or an empty set, and $(L_1 \cap \Gamma(q_2)) \backslash \Gamma(q_1)$ contains a vertex, say p. If $q_1 \sim q_2$, then $((\Gamma(q_1) \cap L_1) \cup \{q_2, x\}) \subseteq \Gamma(q_1, p)$, a contradiction. If $\Gamma(q_1, q_2) \cap \Delta$ is a 2-clique, then $|\Gamma(q_1, q_2, z_1)| = 1$, contrary to Lemma 3(4). The claim is proved.

Claim There is a vertex $q \in \Gamma(z_1) \cap X$ such that, without loss of generality, $L_1 \subset \Gamma(q)$.

Since the graph Δ is regular with valency 10 whereas the graph $\Gamma(x)$ is regular with valency 16, we see that each vertex $p \in \Delta$ is adjacent to 3 vertices of $\Gamma(x, z_1)$ exactly (so that $\Gamma(z_1, p) \subset X \cup \{x\}$). Hence there are 60 edges between the sets Δ and $\Gamma(z_1) \cap X$. On the other hand, $\Gamma(z_1) \cap X$ contains 14 vertices and, therefore, there is a vertex $q \in \Gamma(z_1) \cap X$ such that $|\Gamma(q) \cap \Delta| \geq 5$. In particular, we may assume that $|\Gamma(q) \cap L_1| \geq 3$.

If $L_1 \not\subset \Gamma(q)$, then $\Gamma(q) \cap \Delta$ is a (2×3)-grid. Now there are 54 edges between the sets Δ and $(\Gamma(z_1) \cap X) \backslash \{q\}$. On the other hand, $(\Gamma(z_1) \cap X) \backslash \{q\}$ contains 13 vertices and, hence, there is a vertex $q' \in (\Gamma(z_1) \cap X) \backslash \{q\}$ such that $|\Gamma(q') \cap \Delta| \geq 5$. Suppose that the subgraph $\Gamma(q') \cap \Delta$ is a (2×3)-grid. Then $q \not\sim q'$ and $\Gamma(q, q') \cap \Delta$ is an empty set.

Further, there are 48 edges between the sets Δ and $(\Gamma(z_1) \cap X) \backslash \{q, q'\}$. On the other hand, $(\Gamma(z_1) \cap X) \backslash \{q, q'\}$ contains 12 vertices, and there is a vertex $q'' \in (\Gamma(z_1) \cap X) \backslash \{q, q'\}$ such that $|\Gamma(q'') \cap \Delta| \geq 4$. If $\Gamma(q'') \cap \Delta$ is a (2×3)-grid or (2×2)-grid, then the vertices z_2, q, q', q'' and a vertex of $\Delta \backslash (\Gamma(q) \cup \Gamma(q') \cup \Gamma(q''))$ induce a 5-coclique of $\Gamma(x)$, contrary to Lemma 3(2). The claim is proved.

Claim There are three vertices $q_1, q_2, q_3 \in \Gamma(z_1) \cap X$ such that $L_1 \subset \Gamma(q_i)$ holds for all i.

Choose $q_1 \in \Gamma(z_1) \cap X$ with property $L_1 \subset \Gamma(q_1)$. Since $|\Gamma(q_1, z_2, x)| \geq 2$ holds by Lemma 3(4), we see that q_1 has at most 3 neighbors in $\Gamma(z_1) \cap X$. On the other hand, for each vertex $p \in L_1$, we have $\Gamma(z_1, p) \subset X \cup \{x\}$ and $|\Gamma(z_1, p, q_1)| \geq 2$. Therefore, there is a vertex $q_2 \in \Gamma(z_1, q_1) \cap X$ such that $|\Gamma(q_2) \cap L_1| \geq 10/3 > 3$. This implies that $L_1 \subset \Gamma(q_2)$, and $q_1 \sim q_2$.

Finally, for each vertex $p \in L_1$, the subgraph $\Gamma(z_1, p)$ contains the vertices x, q_1, q_2 and one more vertex of $X \cap (\Gamma(q_1) \cup \Gamma(q_2)) \backslash \{q_1, q_2\}$ (and $X \cap (\Gamma(q_1) \cup \Gamma(q_2)) \backslash \{q_1, q_2\}$ contains at most 4 vertices). It follows that there is a vertex $q_3 \in (\Gamma(z_1) \cap X) \backslash \{q_1, q_2\}$ such that $|\Gamma(q_3) \cap L_1| \geq 10/4 > 2$. Since q_3 and $q_i, i = 1, 2$, have at least $5 > c_2$ common neighbors: z_1, x, and at least three ones in L_1, we see that the vertices q_1, q_2, q_3 are mutually adjacent. If $L_1 \not\subset \Gamma(q_3)$, then, for each vertex $p \in L_1 \backslash \Gamma(q_3)$, the vertices p and q_3 have at least $6 > c_2$ common neighbors: q_1, q_2, x, and at least three ones in $L_1 \cap \Gamma(q_3)$, a contradiction. This yields that $L_1 \subset \Gamma(q_3)$ and $\{x, q_1, q_2, q_3\} \cup L_1$ induces a 14-clique of Γ. The lemma is proved. □

Proof Let us prove Theorem 1. It follows from Lemmas 3(1) and 5 that Γ is a Terwilliger graph. This is impossible by [1, Corollary 1.16.6]. □

The proof of Theorem 2 is quite straightforward. Let Γ is a distance-regular graph with intersection array $\{69, 48, 24; 1, 4, 46\}$. Then Γ has spectrum $69^1, 23^{133}, -1^{874}, -7^{322}$. Choose the vertices $x, y \in \Gamma$ at distance 2. The lemma below follows by the same arguments as in the proof of Lemma 3.

Lemma 6 *The following holds.*

(1) *a clique of* Γ *contains at most* 10 *vertices*;
(2) $\Gamma(x)$ *does not contain a* 5-*coclique*;
(3) $\Gamma(x, y)$ *does not contain a* 3-*coclique.*

Let us show Theorem 2. Note that by [1, Corollary 1.16.6] Γ is not a Terwilliger graph. Hence $\Gamma(x, y)$ contains a 2-coclique and, by Lemma 6(2), the subgraph $\Delta := \Gamma(x) \cap \Gamma_3(y)$ does not contain a 3-coclique. Therefore, for a vertex $z \in \Delta$, the subgraph $\Delta \backslash \Delta(z)$ is a clique. By Lemma 6(1), we have $|\Delta(z)| \geq 14$ and hence, for a pair of non-adjacent vertices $z, z' \in \Delta$, $\Delta(z, z') \geq 6 > c_2$ holds, which is impossible. □

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