Cameron – Liebler line classes in $PG(3, q)$

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based on joint work with **Ivan Mogilnykh**,
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An elementary question

Call a matrix (resp. vector) a $(0,1)$-matrix (resp., a $(0,1)$-vector) if all of its entries are 0 or 1.

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Question 1

Let A be a $(0,1)$-matrix of size $m \times n$ $(n \geq m)$ over the rational numbers $\mathbb{Q}$.
Which vectors in the row space of A are $(0,1)$-vectors?

The question is of particular interest if A is the incidence matrix of a 2-design D (in this case the rank of A is m).

Designs

A 2-design with parameters (v, k, λ) is a pair $D = (X, \mathcal{B})$:

- X is a v-set (with elements called points),
- $\mathcal{B}$ is a collection of k-subsets of X (called blocks),
- every 2 distinct points belong to precisely λ blocks.

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For a 2-design $D = (X, \mathcal{B})$:

$$|\mathcal{B}| \geqslant |X|.$$

(Fisher's inequality)

D is symmetric if $|\mathcal{B}| = |X|$.

Designs, incidence matrix

The incidence matrix of D:

$$A := \begin{array}{cc} & \mathcal{B} \\ & B \\ X, \quad p & \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & A_{p,B} = \begin{cases} 1 \text{ if } p \in B, \\ 0 \text{ if } p \notin B. \end{cases} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \end{array}$$

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The columns of A correspond to the blocks of D, so a $(0,1)$-vector of the row space can be considered as the characteristic function of a set $\mathcal{L}$ of blocks of D.

Sets $\mathcal{L}$ that arise in this way may have very interesting properties.

Designs, automorphisms

An automorphism (or a collineation) of D: (γ, δ)

$$\gamma : X \to X,\ \delta : \mathcal{B} \to \mathcal{B} \text{ such that}$$
$$p \in B \ \Leftrightarrow\ \gamma(p) \in \delta(B) \text{ for all } p \in X, B \in \mathcal{B}.$$

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Consider a group $G \leqslant \mathrm{Aut}(D)$ and its orbits on X and $\mathcal{B}$:

$\mathcal{B}$

		↺	↺	↺
	↻	…	…	…
X	↻	…	incidence matrix	…
	↻	…	…	…

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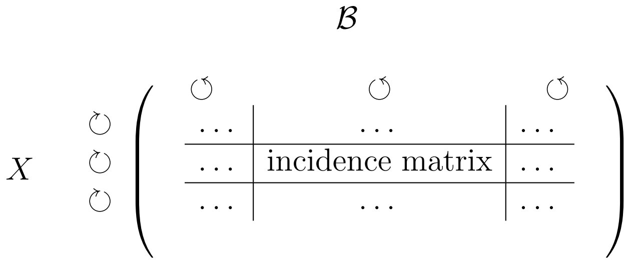

Then

$$\#\{\text{orbits on } \mathcal{B}\} \geqslant \#\{\text{orbits on } X\}.$$

(Block's Lemma)

Designs, tactical decomposition

Consider a decomposition of D:

$$X = X_1 \dot{\cup} \ldots \dot{\cup} X_s,\ \mathcal{B} = \mathcal{L}_1 \dot{\cup} \ldots \dot{\cup} \mathcal{L}_t$$

$$\begin{array}{c} \\ X_1 \\ X_{\ldots} \\ X_s \end{array} \begin{array}{c} \begin{array}{ccc} \mathcal{L}_1 & \mathcal{L}_{\ldots} & \mathcal{L}_t \end{array} \\ \left(\begin{array}{c|c|c} \cdots : & \ldots \vdots & \ddots \\ \hline .. : & \text{incidence matrix} & \ldots \vdots \\ \hline \cdots : & \ddots & \cdots : \end{array} \right) \end{array}$$

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This decomposition is called block-tactical if there is an $s \times t$ matrix $P = (p_{ij})$ such that every block of $\mathcal{L}_j$ contains exactly p_{ij} points of X_i. In other words, the incidence matrix $(X_i, \mathcal{L}_j)$ has constant column sum for all i, j

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Lemma (Block)

Suppose the incidence matrix of D has rank equal to the number of points. Then, for any block-tactical decomposition, the matrix P has rank s, and in particular $t \geq s$.

Designs, tactical decomposition

A tactical decomposition $\mathcal{T}$ of D:

$$X = X_1 \dot\cup \ldots \dot\cup X_s,\ \mathcal{B} = \mathcal{L}_1 \dot\cup \ldots \dot\cup \mathcal{L}_t$$

such that the incidence matrix $(X_i, \mathcal{L}_j)$ has constant row and column sums for all i, j.

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Then

$$t \geqslant s.$$

$\mathcal{T}$ is symmetric if $t = s$.

Designs, tactical decomposition

Proposition (Cameron & Liebler, 1982)

Suppose D is a 2-design. Then a block-tactical decomposition with equally many point and block classes is tactical (and, clearly, symmetric).

Projective geometry

Let $V = GF(q)^{n+1}$.

- $PG(n,q)$ — n-dim. projective space over $GF(q)$

$$\sim : \mathbf{x} \sim \mathbf{y} \ (\mathbf{x}, \mathbf{y} \in V \setminus \{\mathbf{0}\}) \Leftrightarrow \exists \alpha \in GF(q) : \mathbf{x} = \alpha \mathbf{y}$$

$$PG(n,q) = (V \setminus \{\mathbf{0}\}) / \sim$$

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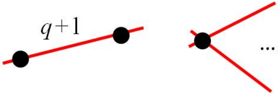
  $$PG(n,q) = (V \setminus \{\mathbf{0}\}) / \sim$$

- point of $PG(n,q)$
  1-dim. vector subspace of V
- line of $PG(n,q)$
  2-dim. vector subspace of V
- ...
- hyperplane
  n-dim. vector subspace of V
- spread — a line set partitioning the points of $PG(n,q)$

Projective geometry $PG(3, q)$

- $\exists!$ line through $\forall$ pair of points with exactly $q + 1$ points on a line, while $\forall$ pair of lines has at most one point in common



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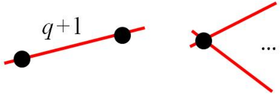
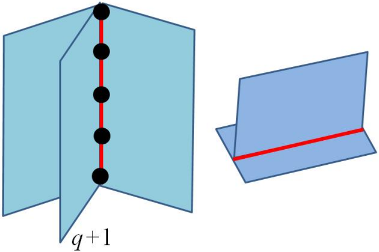


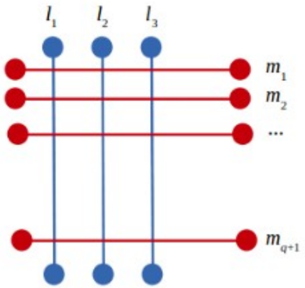
- a line belongs to exactly $q+1$ planes, and $\exists!$ line in the intersection of $\forall$ pair of planes



- the 'point-plane' duality (a plane is a hyperplane)

Projective geometry $PG(3, q)$

- let l_1, l_2, l_3 be any three pairwise skew lines. Then there exist exactly $q + 1$ pairwise skew lines $m_1, \ldots, m_{q+1}$ such that l_i meet m_j for any i and j.



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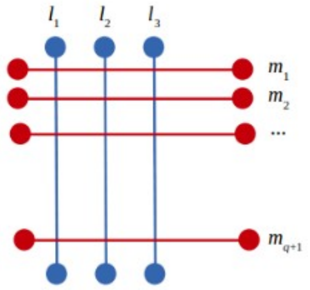

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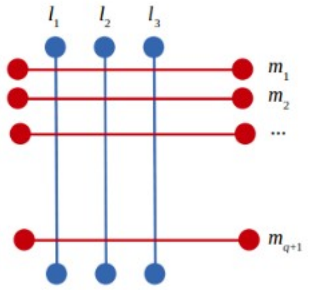

- the set $m_1, \ldots, m_{q+1}$ is determined uniquely, and is called a regulus.
- the set l_1, l_2, l_3 can be uniquely extended to the set of pairwise skew lines $l_1, l_2, l_3, \ldots, l_{q+1}$ such that l_i meet m_j for any i and j (opposite regulus).

Collineations of $PG(n, q)$

A collineation of $PG(n, q)$ is a bijection on the points that preserves incidence.

$\mathrm{Aut}(PG(n, q))$ – the collineation group:

$$P\Gamma L(n+1, q) = PGL(n+1, q) \rtimes \mathrm{Aut}(GF(q)).$$

(The fundamental theorem of p. g.)

Projective geometry as a design

For an arbitrary $1 \leqslant l \leqslant n-1$ let

$\mathcal{B}$ be the set of all l-dim. subspaces of $PG(n,q)$, and

X be the set of points of $PG(n,q)$.

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Then $D = (X, \mathcal{B})$ is a 2-design with incidence = inclusion.

We will be interested in the case $l = 1$, i.e., D will be the design on points and lines of $PG(n,q)$ with $\mathrm{Aut}(D) = P\Gamma L(n,q)$ (the point-line design of $PG(n,q)$).

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- $n = 2$: D is a symmetric design (projective plane).
  $|X| = |\mathcal{B}|$
  $|\{\text{orbits on } X\}| = |\{\text{orbits on } \mathcal{B}\}|\ \forall\ G \leqslant P\Gamma L(3, q)$
  $t = s$ for $\forall$ tactical decomposition $\mathcal{T}$.

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  $t = s$ for $\forall$ tactical decomposition $\mathcal{T}$.
- $n > 2$: **?**

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Conjecture on groups (Cameron, Liebler, 1982)

Such a group is:

- line-transitive

  or
- fixes a hyperplane and acts line-transitive on it

  or (dually)
- fixes a point and acts line-transitive on lines through it.

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The only orbit case:
If $G \leqslant \mathrm{Aut}(D)$ is line-transitive $\Rightarrow$ point-transitive.

Classification of line-transitive subgroups of $P\Gamma L(n,q)$

Cameron, Kantor (1979).

Combinatorial question

Let $n \geq 3$.
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Symmetric t. d. of the point-line design of $PG(n, q)$
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$$\downarrow \quad \not\uparrow$$

Every line class $\mathcal{L}$ is **'special'**

Cameron – Liebler line class (due to Penttila)

(Cameron, Liebler)

Combinatorial question: reduction to $n = 3$

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Let P be the decomposition matrix of s.t.d. of the point-line design of $PG(n, q)$, W be any 3-space of $PG(n, q)$.

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Let P be the decomposition matrix of s.t.d. of the point-line design of $PG(n,q)$, W be any 3-space of $PG(n,q)$.

The induced decomposition of W is given by $X_i \cap \text{points}(W)$ and $\mathcal{L}_j \cap \text{lines}(W)$ for all i, j.

Observe that the induced decomposition is block-tactical, since for any line of W all its points belong to W.

Combinatorial question: reduction to $n = 3$

We can write the decomposition matrix P so that the line classes represented in W come first:

$$P = \begin{pmatrix} P_1 & Q \\ 0 & R \end{pmatrix},$$

where 0 because if line class $\mathcal{L}_j$ is represented in W, then so are all the points classes which meet lines of $\mathcal{L}_j$.

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Suppose that P_1 has size $s_0 \times t_0$. Then $t_0 \geq s_0$ by Block's Lemma, as the induced decomposition is block-tactical.

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Therefore $t_0 = s_0$, the induced decomposition is symmetric and block-tactical and, hence, tactical, by Proposition of Cameron & Liebler.

Properties of a Cameron – Liebler line class $\mathcal{L}$

In what follows let $n = 3$ and $\mathcal{L}$ be a line class in a symmetric t. d. $\mathcal{T}$ of D, point-line design of $PG(3, q)$.

Properties of a Cameron – Liebler line class $\mathcal{L}$, 1

$\exists$ a number x: for $\forall$ spread S

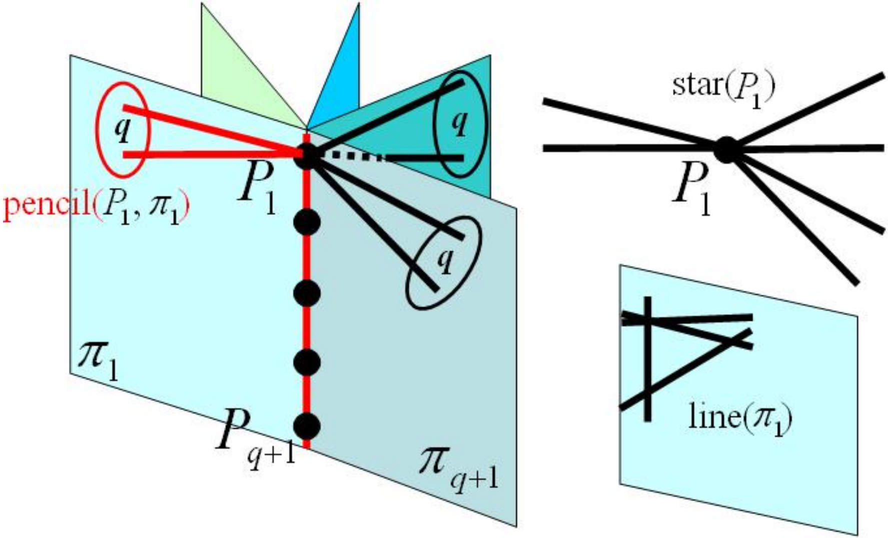
$$|\mathcal{L} \cap S| = x$$

- spread — a line set partitioning the points of $PG(n, q)$

Properties of a Cameron – Liebler line class $\mathcal{L}$, 2

$\exists$ a number x: for $\forall$ point P and $\forall$ plane π with $P \in \pi$:

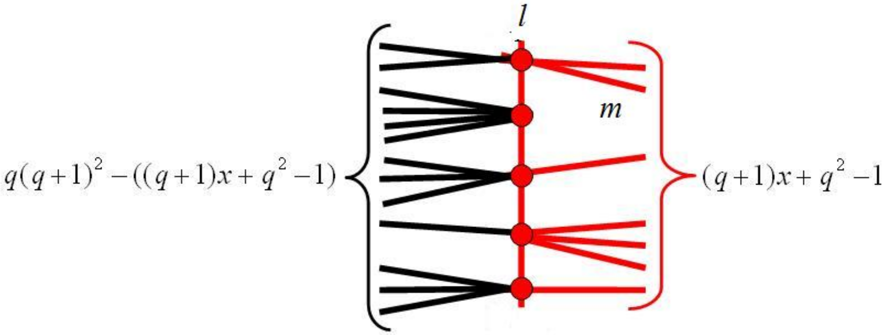
$$|\text{star}(P) \cap \mathcal{L}| + |\text{line}(\pi) \cap \mathcal{L}| = x + (q+1)|\text{pencil}(P,\pi) \cap \mathcal{L}|$$



Properties of a Cameron – Liebler line class $\mathcal{L}$, 3

$\exists$ a number x: $\forall$ line $l \in \mathcal{L}$

$$|\{m \in \mathcal{L} :\ m \text{ meets } l,\ m \neq l\}| = (q+1)x + q^2 - 1$$



Properties of a Cameron – Liebler line class $\mathcal{L}$, 4

$\exists$ a number x: for $\forall$ skew lines l, m

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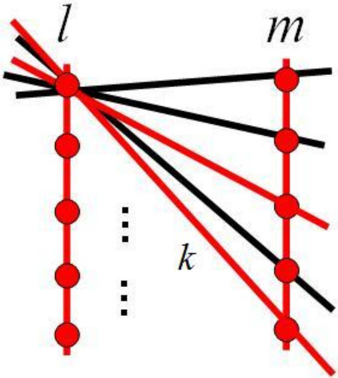

Properties of a Cameron – Liebler line class $\mathcal{L}$, 5

for every regulus $\mathcal{R}$ and its opposite, $\mathcal{R}^{opp}$,

$$|\mathcal{R} \cap \mathcal{L}| = |\mathcal{R}^{opp} \cap \mathcal{L}|$$

Properties of a Cameron – Liebler line class

In a summary, if $\mathcal{L}$ is a line class in a symmetric t. d. of D:

- there exists a number x s.t. $|\mathcal{L} \cap S| = x$ for $\forall$ spread S.
- there exists a number x s.t.

$$|\text{star}(P) \cap \mathcal{L}| + |\text{line}(\pi) \cap \mathcal{L}| = x + (q+1)|\text{pencil}(P, \pi) \cap \mathcal{L}|$$

- there exists a number x s.t. $\forall$ line $l \in \mathcal{L}$

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- there exists a number x s.t. for $\forall$ skew lines l, m

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- for every regulus $\mathcal{R}$ and its opposite, $\mathcal{R}^{opp}$,

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$$|\{m \in \mathcal{L} : \ m \text{ meets } l, \ m \neq l\}| = (q+1)x + q^2 - 1$$

- there exists a number x s.t. for $\forall$ skew lines l, m

$$|\{k \in \mathcal{L} : \ k \text{ meets } l \ \& \ m\}| = x + 2q$$

- for every regulus $\mathcal{R}$ and its opposite, $\mathcal{R}^{opp}$,

$$|\mathcal{R} \cap \mathcal{L}| = |\mathcal{R}^{opp} \cap \mathcal{L}|.$$

x – the same in each of the properties – the parameter of $\mathcal{L}$.
$|\mathcal{L}| = x(q^2 + q + 1)$ ($\Rightarrow x \leqslant q^2 + 1$).

(Cameron, Liebler; Penttila)

Cameron – Liebler line classes, examples

Any line class that satisfies the properties above is called *Cameron – Liebler* line class.

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A line class $\overline{\mathcal{L}}$ complement to $\mathcal{L}$ is also a Cameron – Liebler line class with $x(\overline{\mathcal{L}}) = q^2 + 1 - x(\mathcal{L}) \Rightarrow$ w.l.o.g. $x \leqslant \frac{q^2+1}{2}$

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Conjecture on 'special' classes (Cameron, Liebler)

The only Cameron – Liebler line classes are those listed above (i.e., $x \notin \{3, \ldots, q^2 - 2\}$?).

Cameron – Liebler line classes and Question 1

Consider the incidence matrix of D:

$$A := \quad \begin{matrix} & lines, \mathcal{B} \\ points, X & \begin{pmatrix} 0 & 1 & \dots \\ 0 & 0 & \dots \\ \dots & \dots & \dots \end{pmatrix} \end{matrix}$$

The incidence map associated with D is the linear map

$$\alpha : \ \mathbb{Q}^X \to \mathbb{Q}^{\mathcal{B}} \text{ and its dual } \alpha^* : \ \mathbb{Q}^{\mathcal{B}} \to \mathbb{Q}^X$$

with $(f\alpha)(B) := \sum_{p \in B} f(p)$ and $(g\alpha^*)(p) = \sum_{B \ni p} g(B)$ for all $B \in \mathcal{B}$, $f \in \mathbb{Q}^X$, $p \in X$, $g \in \mathbb{Q}^{\mathcal{B}}$.

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for all $B \in \mathcal{B}$, $f \in \mathbb{Q}^X$, $p \in X$, $g \in \mathbb{Q}^{\mathcal{B}}$.

Note that

$$\mathbb{Q}^{\mathcal{B}} = \mathrm{im}(\alpha) \oplus \ker(\alpha^*),$$

where this sum is orthogonal with respect to the usual form

$$(f, g) = \sum_{B \in \mathcal{B}} f(B) g(B).$$

Cameron – Liebler line classes and Question 1

Question 1

Let A be a $(0,1)$-matrix of size $|X| \times |\mathcal{B}|$ over the rational numbers $\mathbb{Q}$.

Which vectors in the row space of A are $(0,1)$-vectors?

Cameron – Liebler line classes and Question 1

Question 1

Let A be a $(0,1)$-matrix of size $|X| \times |\mathcal{B}|$ over the rational numbers $\mathbb{Q}$.
Which vectors in the row space of A are $(0,1)$-vectors?

So we want to find a set $\mathcal{L}$ of lines such that its characteristic vector $\chi_{\mathcal{L}}$ is in $\mathrm{im}(\alpha)$.

It turns out that the following two equivalent properties:

- $\chi_{\mathcal{L}} \in \mathrm{im}(\alpha)$,
- $\chi_{\mathcal{L}} \in \ker(\alpha^*)^{\perp}$,

are also equivalent to the properties of Cameron – Liebler line classes!

Cameron& Liebler; Penttila

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Further, since $G := PGL(4, q)$ preserves incidence,

$$\langle \mathbf{e} \rangle,\ \langle \mathbf{e} \rangle^{\perp} \cap \mathrm{im}(\alpha), \text{ and } \ker(\alpha^*)$$

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Since G has rank 3 on lines, they are all irreducible.

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It now follows that $f \in \ker(\alpha^*)^{\perp}$ if, and only if, $f \perp g^{PGL(4,q)}$, for some $g \in \ker(\alpha^*)$.

Cameron – Liebler line classes and Question 1

Proof:
In order to show

- $\chi_{\mathcal{L}} \in \mathrm{im}(\alpha) \Leftrightarrow$ there exists a number x s.t. $\forall$ line $l \in \mathcal{L}$

$$|\{m \in \mathcal{L} : \ m \text{ meets } l, \ m \neq l\}| = (q+1)x + q^2 - 1,$$

we take g to be

$$\chi_M - (q^2-1)\chi_l - \tfrac{q+1}{q^2+q+1}\mathbf{e},$$

where $M := \{m : \ m \text{ meets } l, \ m \neq l\}$
(it is an easy exercise to show that $g \in \ker(\alpha^*)$).

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(it is an easy exercise to show that $g \in \ker(\alpha^*)$).

As $g \in \ker(\alpha^*)$, we have $(\chi_{\mathcal{L}}, g) = 0$, which gives us the required property from:

$$(\chi_{\mathcal{L}}, \chi_M) - (q^2-1)(\chi_{\mathcal{L}}, \chi_l) - \tfrac{q+1}{q^2+q+1}(\chi_{\mathcal{L}}, \mathbf{e}) = 0.$$

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Proof:
In order to show

- $\chi_{\mathcal{L}} \in \mathrm{im}(\alpha) \Leftrightarrow$ there exists a number x s.t. $|\mathcal{L} \cap S| = x$ for $\forall$ spread S,

we take g to be

$$\chi_S - \frac{1}{q^2+q+1}\mathbf{e}.$$

Indeed, this is in $\ker(\alpha^*)$, as $(\chi_S \alpha^*)(p) = \sum_{B \ni p} \chi_S(B) = 1$, as every point is covered by a unique line.

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which gives the property.

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  (Metsch'10)

Cameron – Liebler line classes, some background

- In 2011 M. Rodgers constructed new Cameron – Liebler line classes for many odd values of q ($q < 200$) satisfying $q \equiv 1 \mod 4$ and $q \equiv 1 \mod 3$, having parameter $x = \frac{1}{2}(q^2 - 1)$.

  These new examples are made up of a union of orbits of a cyclic collineation group having order $q^2 + q + 1$.

Drudge's approach

The most of the previous results rely on the observation by K. Drudge.

Define a clique of $PG(3, q)$ to be a set of the following type: star(P), for some point P, or line(π), for some plane π (reason: $\forall$ two lines in a clique intersect).

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A clique $\mathcal{C}$ of $PG(3, q)$ and its lines may be considered as a projective plane $PG(2, q)$ and its points, resp. (A natural correspondence is provided by the Klein correspondence.)

(Recall also that $PG(2, q)$ contains $q^2 + q + 1$ points.)

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(Recall also that $PG(2, q)$ contains $q^2 + q + 1$ points.)

A blocking set in $PG(2, q)$ is a set of points that intersects every line but contains no line.

Drudge's approach

Let $\mathcal{L}$ be a Cameron – Liebler line class with parameter x in $PG(3,q)$, $\mathcal{C}$ be a clique, and assume that there exists no CL line class of parameter $x-1$.

This implies that $|\mathcal{C} \cap \mathcal{L}| < q^2+q+1$, because if $\mathcal{C} \subset \mathcal{L}$ then $\mathcal{L} \setminus \mathcal{C}$ would be a line class of parameter $x-1$.

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Lemma (Drudge, 1999)

If $x < |\mathcal{C} \cap \mathcal{L}| \leq x+q$ then the lines of $\mathcal{C} \cap \mathcal{L}$ form a blocking set in $\mathcal{C}$.

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Keep in mind:

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$\nexists$ line class of parameter $x-1 \Rightarrow |\mathcal{C} \cap \mathcal{L}| < q^2+q+1$.

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Let P be a point, and $\pi \ni P$ be a plane. Denote:

$M := |\text{star}(P) \cap \mathcal{L}|$, $N := |\text{line}(\pi) \cap \mathcal{L}|$, $Q := |\text{pencil}(P, \pi) \cap \mathcal{L}|$.

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Then, by the properties, $M+N = x+(q+1)Q$ holds.

- if $Q=0$ and $x<M$, then $N = x+(q+1)Q-M<0$!
- if $Q=q+1$ and $M \leq x+q$, then $N \geq q^2+q+1$!

So we conclude that $0 < Q < q+1$, i.e., $\mathcal{L}$ induces a blocking set in star(P).

Cameron – Liebler line classes in $PG(3,4)$

$x \in \{0!, 1!, 2!, \not 3, \not 4, \not 5, 6?, 7!?, 8?\}$ (as $(q^2+1)/2 = 8.5$)

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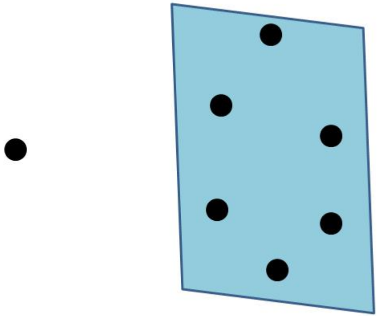
The Govaerts – Penttila construction for $x = 7$, $q = 4$.



hyperoval in $PG(2,q)$ – a set of $q+2$ points, no 3 of which collinear

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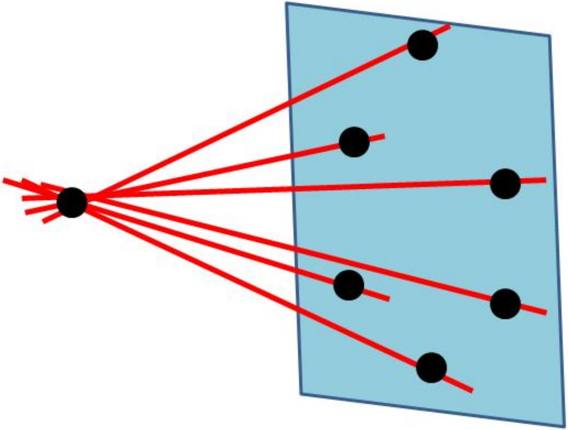

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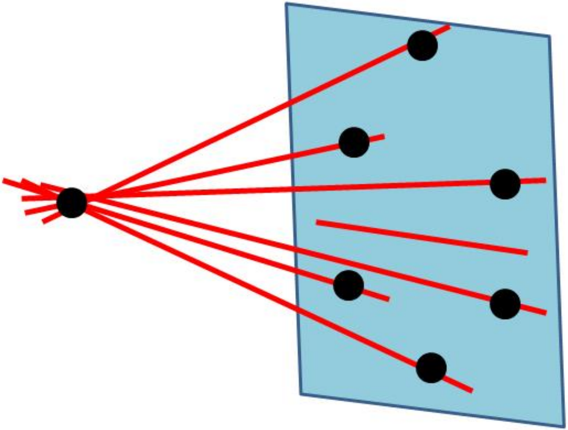

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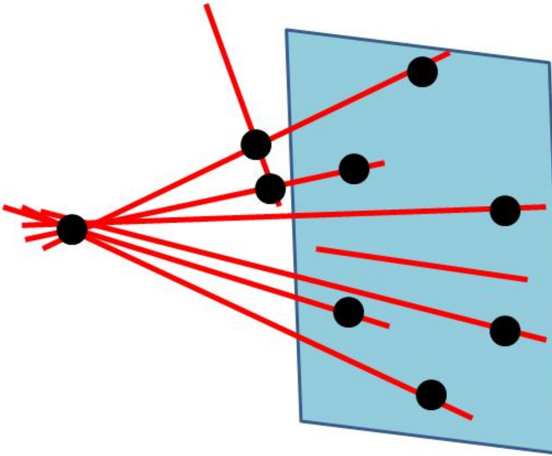
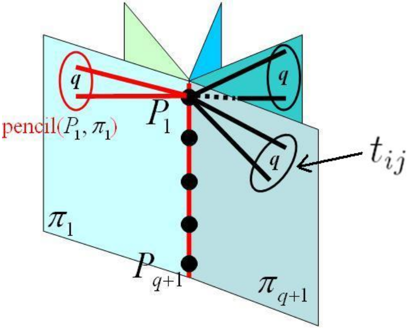


Patterns (G. & Mogilnykh, 2012)

Let l be a line of $PG(3,q)$, $\mathcal{L}$ a Cameron – Liebler line class. Consider all the points P_i, $i = 1, \ldots, q+1$ that are on l, and all the planes π_j, $j = 1, \ldots, q+1$ that contain l.



Define a square matrix T of order $q+1$ whose (i,j)-element is $|\mathrm{pencil}(P_i, \pi_j) \cap \mathcal{L} \setminus \{l\}|$

We will call such matrix pattern w.r.t. l.

Properties of patterns

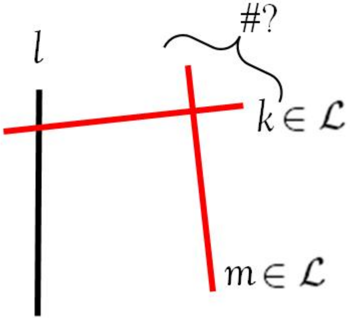
Let $T := (t_{ij})$ be a pattern w.r.t. a line l, and $\chi := 0$ if $l \notin \mathcal{L}$, and 1 otherwise. Then the following hold:

- $0 \leq t_{ij} \leq q$ for all $i, j \in \{1, \ldots, q+1\}$;
- $\displaystyle\sum_{i,j=1}^{q+1} t_{ij} = x(q+1) + \chi(q^2 - 1)$;
- $\displaystyle\sum_{j=1}^{q+1} t_{kj} + \sum_{i=1}^{q+1} t_{il} = x + (q+1)(t_{kl} + \chi), \forall k, l$;
- $\displaystyle\sum_{i,j=1}^{q+1} t_{ij}^2 = (x - \chi)^2 + q(x - \chi) + \chi q^2(q+1).$

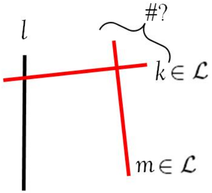
Properties of patterns

$$\sum_{i,j=1}^{q+1} t_{ij}^2 = (x-\chi)^2 + q(x-\chi) + \chi q^2(q+1)$$

follows from the two-side counting of $(k,m) \in \mathcal{L} \times \mathcal{L}$ such that $l \sim k$, $k \sim m$ and $l \nsim m$.



Properties of patterns: sketch of proof



Fix $l \in \mathcal{L}$ and $m \in \mathcal{L}$ such that l and m are skew.
Then, by the property for skew lines of $\mathcal{L}$,

$$\mu(l, m) := |\{k \in \mathcal{L} : \ k \text{ meets } l \ \& \ m\}| = x + 2q,$$

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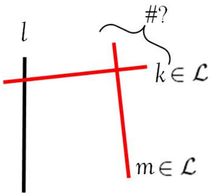

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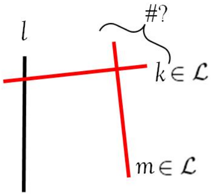
so we can calculate explicitly the number N of such pairs (k,m), i.e.:

$$N = \sum_{m \in \mathcal{L}, \ m \text{ skew to } l} \mu(l,m).$$

(as we also know the number of lines $m \in \mathcal{L}$, skew to l.)

Properties of patterns: sketch of proof



Fix now $l \in \mathcal{L}$ and $k \in \mathcal{L}$ such that $l \cap k \neq \emptyset$.
Then it follows from the 'star-line-pencil' property, that

$\lambda(l,k) := |\{m \in \mathcal{L} : \ m \text{ meets } k, \text{ skew to } l\}| = q(x+q-t-1)$

where $t := |\text{pencil}(P,\pi) \cap \mathcal{L} \setminus \{l\}|$ and $\text{pencil}(P,\pi)$ contains a line k, i.e., $P \in k, l \in \pi$.

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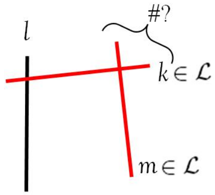

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$\lambda(l,k) := |\{m \in \mathcal{L} : \ m \text{ meets } k, \text{ skew to } l\}| = q(x+q-t-1)$
where $t := |\text{pencil}(P,\pi) \cap \mathcal{L} \setminus \{l\}|$ and $\text{pencil}(P,\pi)$ contains a line k, i.e., $P \in k, l \in \pi$.

So we can calculate the number N of such pairs (k,m), i.e.:

$$N = \sum_{k \in \mathcal{L},\ k \text{ meets } l} \lambda(l,k),$$

and this gives the sum in terms of t_{ij}^2.

Cameron–Liebler line classes in $PG(3, 4)$

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Let $x=7$. We have only the following admissible patterns:

w.r.t. $l \in \mathcal{L}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 4 & 2 & 3 & 2 \\ 4 & 4 & 2 & 3 & 2 \\ 3 & 3 & 1 & 2 & 1 \\ 2 & 2 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \end{pmatrix}$$

w.r.t. $l \notin \mathcal{L}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 4 & 3 & 3 & 3 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 2 & 2 \\ 3 & 2 & 2 & 2 & 2 \\ 3 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

Cameron–Liebler line classes in $PG(3,4)$, $x = 7$

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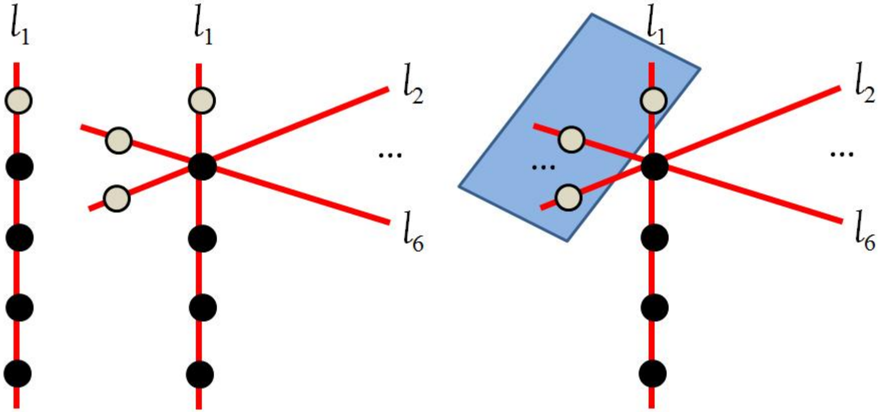
Cameron–Liebler line classes in $PG(3,4)$, $x=7$



Cameron–Liebler line classes in $PG(n, 4)$

Let $\mathcal{L}$ be a Cameron – Liebler line class in $PG(n, q)$, $n > 3$, X be a 3-dim. subspace of $PG(n, q)$.

- line$(X) \cap \mathcal{L}$ is a Cameron – Liebler line class of $X = PG(3, q)$.
- if $\mathcal{L}$ is known $\Rightarrow$ line$(X) \cap \mathcal{L}$ is known.

(Drudge)

Summary of Part I

Accepted in Designs, Codes and Cryptography (2013?)

- a new existence condition for Cameron – Liebler line classes,
- $x \in \{0!, 1!, 2!, \not{3}, \not{4}, \not{5}, \not{6}, 7!, \not{8}\}$ in $PG(3,4)$
- (complete classification of those classes in $PG(3,4) \Rightarrow PG(n,4)) \Leftrightarrow$ (classification of completely regular codes with strength 0 of the Grassmann graphs $G_4(n+1,2)$).

Conjecture (G., Mogilnykh)

The new existence condition eliminates about a half of possible values of x.

Conjecture

Theorem (G., Metsch)

Suppose $PG(3, q)$, q odd, has a Cameron – Liebler line class with parameter x. Then

$$x(x-1) + 2n(n-x) \equiv 0 \bmod 2(q+1)$$

for some $n \in \{0, \ldots, q\}$.

Conjecture: proof

$$x(x-1)+2n(n-x)\equiv 0 \bmod 2(q+1)$$

Proof:
For a line not from a Cameron – Liebler line class, we have the following condition for its pattern:

$$\sum_{i,j=0}^{q} t_{ij}^2 = x(q+x).$$

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$$\sum_{i,j=0}^{q} t_{ij}^2 = x(q+x).$$

Denote $s := t_{00}$, $e_i := t_{i0}$, and $f_j = t_{0j}$, for $i, j = 1, \ldots, q$. Then $t_{ij} = e_i + f_j - s$ for $i, j \geq 1$ and thus

$$s^2 + \sum_{i=1}^{q} e_i^2 + \sum_{j=1}^{q} f_j^2 + \sum_{i,j=1}^{q} (e_i + f_j - s)^2 = x(q+x).$$

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Define $E := \sum_{i=1}^{q} e_i$ and $F := \sum_{j=1}^{q} f_j$. Then the last equation can be written as

$$(q^2+1)s^2 + (q+1)(\sum_{i=1}^{q} e_i^2 + \sum_{j=1}^{q} f_j^2) + 2EF - 2sq(E+F) = x(q+x).$$

Modulo 2 we have $\sum_i e_i(e_i - 1) \equiv 0$, so $\sum_i e_i^2 \equiv \sum_i e_i \equiv E$.

Conjecture: proof

$$x(x-1)+2n(n-x) \equiv 0 \bmod 2(q+1)$$

Proof:
Hence

$$(q^2+1)s^2+(q+1)(E+F)+2EF+2s(E+F) \equiv \\ x(q+x) \bmod 2(q+1).$$

Conjecture: proof

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Proof:
Hence

$$(q^2+1)s^2+(q+1)(E+F)+2EF+2s(E+F) \equiv x(q+x) \bmod 2(q+1).$$

We also have that $(E+s)+(F+s)=s(q+1)+x$, i.e., $E+F=s(q-1)+x$ (by 'star-line-pencil').

Using this and after some manipulations, we can get the required with $n=E+s$.

Conjecture: proof

Consider now the prime factorization $2(q+1) = p_1^{k_1} \cdots p_s^{k_s}$.
Then, for a given n, the equation

$$x(x-1) + 2n(n-x) \equiv 0 \bmod 2(q+1)$$

is equivalent to

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for $i = 1, \ldots, s$.

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Lemma

For an odd prime p, the number of integers x with $0 \leq x < p$ such that there exists an integer n satisfying

$$x(x-1) + 2n(n-x) \equiv 0 \bmod p$$

is $\frac{1}{2}(p+1)$ when $p \equiv 1 \bmod 4$ and $\frac{1}{2}(p+3)$ when $p \equiv 3 \bmod 4$.

Conjecture: proof

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Proof:

Consider the equation over the finite field $GF(p)$.

As p is odd, we write $x = 2y$ so that the equation can be written as

$$2y(2y-1) + 2n(n-2y) = 0,$$

which is equivalent to

$$(y + \tfrac{p-1}{2})^2 = (\tfrac{p-1}{2})^2 - (n-y)^2.$$

Conjecture: proof

Proof:

$$(y + \tfrac{p-1}{2})^2 = (\tfrac{p-1}{2})^2 - (n-y)^2.$$

We want to find all y that satisfy this equation for some n.
So we can replace the equation by

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The number of feasible y thus depends on the number of squares in $GF(p)$ that can be written as difference of a given square and an arbitrary square n^2. It is well known in number theory that there are $\frac{1}{4}(p+3)$ such squares n^2 when $p \equiv 1 \bmod 4$ and $\frac{1}{4}(p+3)$ such squares when $p \equiv 3 \bmod 4$.

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Each square corresponds to two values of y, except $n^2 = (\frac{p-1}{2})^2$, which corresponds only to one value of y.