On a Characterization of the Grassmann Graphs $J_q(2d, d)$

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The Grassmann graph $J_q(n,d)$

- Let $q \geq 2$ be a prime power, $n \geq d \geq 1$ be integers.
- $J_q(n,d)$ has as vertices all d-dim. subspaces $U \leq \mathbb{F}_q^n$.
- $U_1 \sim U_2$ if $dim(U_1 \cap U_2) = d-1$.
- $J_q(n,d) \cong J_q(n,n-d)$, diameter equals $\min(d, n-d)$.
- Distance-regular graph (DRG, for short).
- Q-polynomial.

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Characterization by $\iota(\Gamma)$

Theorem (K. Metsch, 1995)

The Grassmann graph $J_q(n, d)$ is characterized by its intersection array with the following *possible exceptions*:

- $n = 2d$, $n = 2d \pm 1$,
- $n = 2d \pm 2$ if $q \in \{2, 3\}$,
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An exception, the *twisted Grassmann graph* with parameters as of $J_q(2d \pm 1, d)$ for all q, was constructed by E. van Dam and J.H. Koolen (2004).

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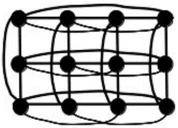
Maximal cliques

$\Gamma_1(x)$ = the **local** graph of vertex x of a graph Γ.

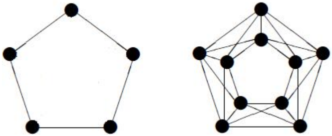
$\Gamma = J_q(n,d)$:

$\Gamma_1(x) = q$-clique extension of $\begin{bmatrix} n-d \\ 1 \end{bmatrix} \times \begin{bmatrix} d \\ 1 \end{bmatrix}$-lattice,

Maximal cliques: $q\begin{bmatrix} n-d \\ 1 \end{bmatrix} + 1$ and $q\begin{bmatrix} d \\ 1 \end{bmatrix} + 1$



3×4-lattice



the 2-clique extension of 5-gon

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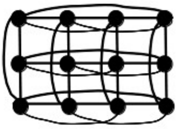

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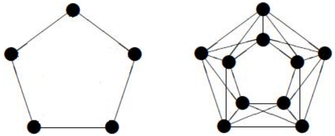

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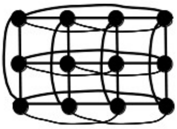

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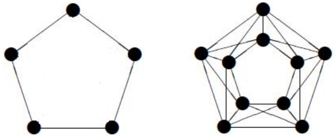

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$\Gamma \to$ partial linear space

Let Γ be $J_q(n, d)$.
Then:

- the set $\mathcal{P}$ of all vertices of Γ,
- the set $\mathcal{L}$ of all maximal cliques of the same size in Γ

form a partial linear space, while Γ is its **point graph**.

Partial linear space is a set $\mathcal{P}$ of *points* and a set $\mathcal{L}$ of *lines* (subsets of $\mathcal{P}$):

- any line contains at least two points;
- any two points are on at most one line;

Now suppose Γ has the same intersection array as $J_q(n, d)$.
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Characterization: partial linear space $\to \Gamma$

.. and then recognize this partial linear space.

Theorem (Ray-Chaudhuri & Sprague, 1976)

Let $(\mathcal{P}, \mathcal{L})$ be a partial linear space s.t. for some $q \geq 2$:

- each line has at least $q^2 + q + 1$ points,
- each point is on more than $q + 1$ lines,
- if $p \in \mathcal{P}$ and $L \in \mathcal{L}$ such that $d(p, L) = 1$, then there are exactly $q + 1$ lines on p meeting L,
- if $p, p' \in \mathcal{P}$ are at distance 2, then there are exactly $q + 1$ lines on p such that $d(p', L) = 1$,
- the point graph Γ of $(\mathcal{P}, \mathcal{L})$ is connected.

Then $(\mathcal{P}, \mathcal{L}, \in) \cong \left(\begin{bmatrix} V \\ d \end{bmatrix}, \begin{bmatrix} V \\ d+1 \end{bmatrix}, \subset\right)$ for a vector space $V = \mathbb{F}_q^n$, and $\Gamma \cong J_q(n, d)$.

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$\iota(\Gamma) \to_? $ partial linear space

The Bruck/Bose-Laskar/Metsch argument

Suppose that Γ is a graph, regular with valency k and:

- for any pair of adjacent vertices, the number of its common neighbours $= \lambda$ (relatively *large*),
- for any pair of non-adjacent vertices, the number of its common neighbours $\leq \mu$ (relatively *small*),
- the valency k is bounded above in terms of λ and μ.

Then Γ contains a set of **grand** (large maximal) cliques s.t.

- each pair of adjacent vertices is contained in exactly one grand clique ($\to$ **line**),
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In terms of intersection numbers:

$$k = b_0$$
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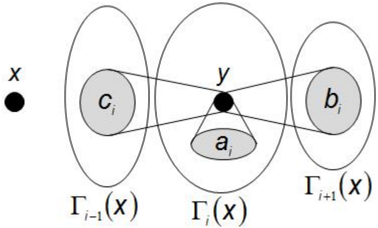

Unfortunately, this purely combinatorial argument does not work for $J_q(n,d)$, if $n \simeq 2d$:

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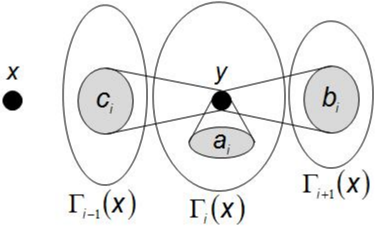

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The Terwilliger polynomial

- Let Γ be a Q-polynomial DRG with diameter $D \geq 3$.
- Terwilliger (early 1990's):
  $\exists$ a polynomial p_T of degree 4 such that, for any vertex $x \in \Gamma$, and any non-principal eigenvalue η of $\Gamma_1(x)$ (*a local eigenvalue at* x) we have $p_T(\eta) \geq 0$.
- p_T only depends on the intersection numbers of Γ and the Q-polynomial ordering of primitive idempotents of its Bose-Mesner algebra.
- We call p_T the **Terwilliger polynomial**.

See:

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Suppose a DRG Γ has the intersection array of $J_q(2d, d)$.

- Its Terwilliger polynomial p_T has the three distinct roots:

$$-q-1, \quad -1, \text{ and } \frac{q^2(q^{d-1}-1)}{q-1} - 1 \text{ (of multiplicity 2)},$$

while the leading term coefficient of p_T is negative.

- $p_T(\eta) \geq 0 \Rightarrow$ a local non-principal eigenvalue η at any vertex $x \in \Gamma$ satisfies:

$$\eta \in [-q-1, -1] \text{ or } \eta = \frac{q^2(q^{d-1}-1)}{q-1} - 1 =: \hat{\theta}_d.$$

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Spectrum of $\Gamma_1(x)$

Suppose that $\Gamma_1(x)$ has spectrum

$$(-q-1)^{g_1}, \quad \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m, \quad (-1)^{g_2}, \quad (\hat{\theta}_d)^{g_3}, \quad (a_1)^1,$$

where $-q-1 < \varepsilon_i < -1$ and a_1 is the valency of $\Gamma_1(x)$.

Let Δ be the q-clique extension of the $\begin{bmatrix} d \\ 1 \end{bmatrix} \times \begin{bmatrix} d \\ 1 \end{bmatrix}$-lattice, i.e., $\Delta \simeq$ the local graph of $J_q(2d, d)$.
Then Δ has spectrum

$$(-q-1)^{f_1}, \quad (-1)^{f_2}, \quad (\hat{\theta}_d)^{f_3}, \quad (a_1)^1,$$

for some multiplicities f_1, f_2, f_3 depending on q and d.
We claim that $\Gamma_1(x)$ has the same spectrum as Δ, i.e.,

$$\{\varepsilon_1, \ldots, \varepsilon_m\} = \emptyset,$$

and $f_1 = g_1$, $f_2 = g_2$, $f_3 = g_3$.

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Note that, by [BCN, Thm 4.4.4], $g_1 \geq f_1$.

Since $\Gamma_1(x)$ and Δ have the same number b_0 of vertices and valency a_1, we have the standard equations:

$$\sum_{i=1}^{3} g_i + m + 1 = \sum_{i=1}^{3} f_i + 1 = b_0,$$

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Local characterization of $J_q(n, d)$

Proposition (G., Koolen)

Let Γ be a DRG with the intersection array of $J_q(2d, d)$, $d \geq 3$. Then, for every vertex $x \in \Gamma$, $\Gamma_1(x)$ has the spectrum of the q-clique extension of the $\begin{bmatrix} d \\ 1 \end{bmatrix} \times \begin{bmatrix} d \\ 1 \end{bmatrix}$-lattice.

This result gives a very strong evidence that $J_q(2d, d)$ is unique.

Local characterization (Numata, Cohen, [BCN, Thm 9.3.8])

$\Gamma \simeq J_q(n, d)$ if $\Gamma_1(x)$ is the q-clique extension of lattice and every μ-graph is the $(q+1) \times (q+1)$-lattice.

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A spectral characterization of the q-clique extension of $r \times s$-lattice.

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Triple intersection numbers

Let Γ be a Q-polynomial DRG with diameter ≥ 3.
Fix a triple of vertices x, y, z such that $x \sim y$, $x \sim z$.

An important idea leading to the Terwilliger polynomial:

$$|\Gamma_i(x) \cap \Gamma_{i+1}(y) \cap \Gamma_{i+1}(z)| = \alpha_{i,\delta}|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_1(z)| + \beta_{i,\delta}$$

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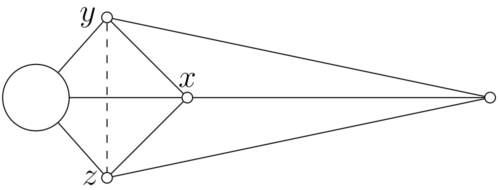

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If $\iota(\Gamma) = \iota(J_q(2d, d))$ and the diameter d is **odd**, then α turns to be non-integer.

One can show that

$$y \not\sim z:$$
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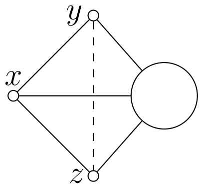

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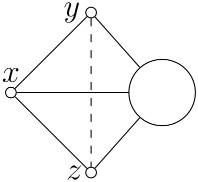

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$\Gamma_1(x)$ is regular and has the four distinct eigenvalues $\Rightarrow$ $\Gamma_1(x)$ is walk-regular, we may compute the number of triangles and quadrangles through any vertex of $\Gamma_1(x)$.

Following Van Dam, we consider a system of linear equations involving τ_ℓ, $\ell \in \{1,4,7\}$, as unknowns.

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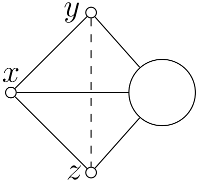

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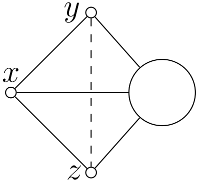

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Summary, open problems

Theorem (G., Koolen)

The Grassmann graph $J_2(2d, d)$ is characterized by its intersection array if d is odd.

	$q = 2$	$q = 3$	$q > 3$	
$2d$	odd $\surd$ / even $\bigstar$	odd $\Diamond\bigstar$ / even $\bigstar$	odd $\Diamond\bigstar$ / even $\bigstar$	the Terwilliger theory
$2d+1$	$\spadesuit$	$\spadesuit$	$\spadesuit$	no one can explain
$2d+2$	odd $\Diamond$ / even ?	odd $\Diamond$ / even ?	$\surd$	
$2d+3$	?	$\surd$	$\surd$	B.-B.-L.-M. argument
$\geq 2d+4$	$\surd$	$\surd$	$\surd$	

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$\bigstar$ — the local graph spectrum is known

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$$p_T(\eta) := p_1(\eta)p_2(\eta) - p_3(\eta)^2,$$

where

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The Terwilliger polynomial

$$p_2(\eta) = p^1_{23}\Big(\tau_0\eta^2 + (\tau_1 - \tau_2)\eta + (1 - a_1\tau_0 - \tau_2)\Big)$$

$$\tau_0 = \frac{1}{b_1} \times \frac{(\theta^*_2 - \theta^*_1)(\theta^*_0 + \theta^*_1 - \theta^*_2 - \theta^*_3)}{(\theta^*_0 - \theta^*_2)(\theta^*_3 - \theta^*_2)},$$

$$\tau_1 = \frac{\theta^*_2 - \theta^*_1}{\theta^*_3 - \theta^*_2}\Big(\frac{(\theta^*_1 - \theta^*_3)}{(\theta^*_0 - \theta^*_2)} - \frac{a_1 - 1}{b_1} - \frac{(a_1 - 1)(\theta^*_1 - \theta^*_3)}{b_1(\theta^*_0 - \theta^*_2)}\Big),$$

$$\tau_2 = \frac{\theta^*_2 - \theta^*_1}{\theta^*_3 - \theta^*_2}\Big(\frac{(\theta^*_1 - \theta^*_3)}{(\theta^*_0 - \theta^*_2)} - \frac{a_1 + 1 - c_2}{b_1} - \frac{(a_1 + 1)(\theta^*_1 - \theta^*_3)}{b_1(\theta^*_0 - \theta^*_2)} +$$
$$+ \frac{(\theta^*_1 - \theta^*_2)^2 - (\theta^*_0 - \theta^*_1)(\theta^*_2 - \theta^*_3)}{(\theta^*_0 - \theta^*_1)(\theta^*_0 - \theta^*_2)} - \frac{1}{b_1} \times \frac{(\theta^*_0 - \theta^*_1)(\theta^*_0 + \theta^*_1 - \theta^*_2 - \theta^*_3)}{(\theta^*_0 - \theta^*_2)(\theta^*_1 - \theta^*_2)}\Big),$$

where

$$E_1 = \theta^*_0 A_0 + \theta^*_1 A_1 + \ldots + \theta^*_D A_D.$$

Thank you!

ありがとうございました