# Perfect 2-colorings of Johnson graphs $J(v, 3)$

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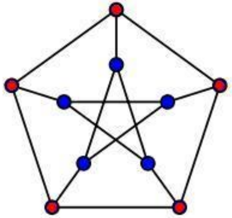
- every vertex of $V_i$ has exactly $p_{ij}$ neighbours of $V_j$,
- $P := (p_{ij})_{t \times t}$ — the *quotient matrix* of coloring.



$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

# Background

- classification of perfect error-correction codes over $\mathbf{F}_q$
  (V. Zinoviev & V. Leontiev'73; A. Tietäväinen'73)
- a notion of *completely regular codes* in association schemes
  (P. Delsarte'73)
- completely regular codes in Hamming and Johnson graphs/schemes
  (P. Delsarte'73)
  (A. Meyerowitz'03)
  (W. Martin'9x)

# Distance partition w.r.t. a code

Let $C \subseteq V(\Gamma)$ be a code in $\Gamma$. For a vertex $x \in V(\Gamma)$, define

$$\text{dist}(x, C) := \min\{\text{dist}(x, y) |\ y \in C\},$$

$\rho_C := \max\{\text{dist}(x, C) |\ x \in V(\Gamma)\}$ – the *covering radius* of $C$.

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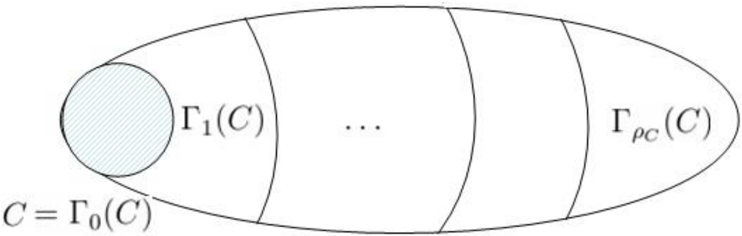
The *distance partition* of $V(\Gamma)$ with respect to $C$:

$$V(\Gamma) = \Gamma_0(C) \cup \Gamma_1(C) \cup \ldots \cup \Gamma_{\rho_C}(C),$$

where $\Gamma_i(C) := \{x \in V(\Gamma) :\ \mathrm{dist}(x, C) = i\}$.



# Completely regular code

A code $C$ is called *completely regular* if the distance partition of $V(\Gamma)$ w.r.t. $C$ is a perfect $(\rho_C + 1)$-coloring:

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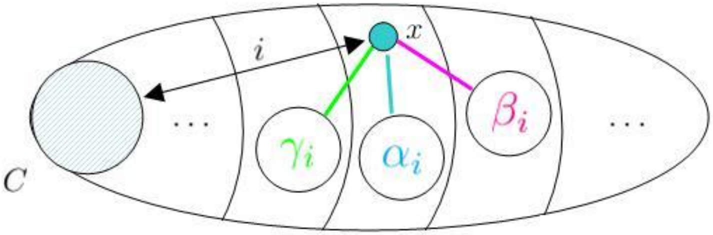
$\gamma_i$ neighbours of $\Gamma_{i-1}(C)$, $\alpha_i$ of $\Gamma_i(C)$, $\beta_i$ of $\Gamma_{i+1}(C)$

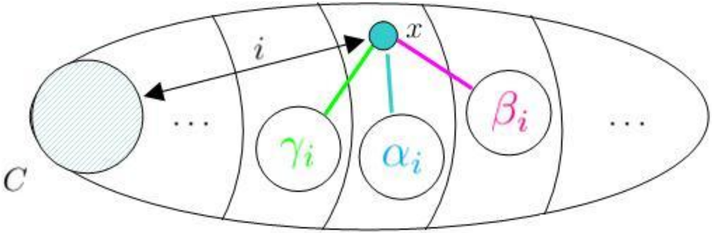



(A. Neumaier'92, P. Delsarte'73)

# Completely regular code

The quotient matrix of this perfect coloring can be written in the following tridiagonal form:

$$P = \begin{pmatrix} \alpha_0 & \beta_0 & 0 & \dots & 0 \\ \gamma_1 & \alpha_1 & \beta_1 & & \\ 0 & \gamma_2 & \alpha_2 & \beta_2 & \\ & & \ddots & \ddots & \ddots \\ 0 & \dots & 0 & \gamma_\rho & \alpha_\rho \end{pmatrix}.$$



# Adjacency and quotient matrices

The adjacency matrix of a graph $\Gamma = (V, E)$:

$$(A)_{x,y} := \begin{cases} 1 \text{ if } \{x, y\} \in E(\Gamma), \\ 0 \text{ if } \{x, y\} \notin E(\Gamma). \end{cases}$$

## Eigenvalues:

Let a graph $\Gamma$ have a perfect $t$-coloring with quotient matrix $P$. Then every eigenvalue of $P$ is an eigenvalue of the adjacency matrix $A$ of $\Gamma$.

# Johnson graph $J(v, k)$

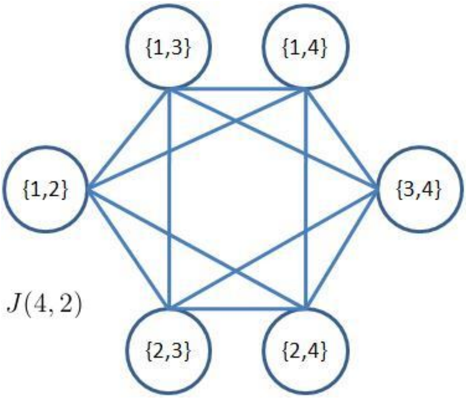
- vertices:

$$V = \{b \subset \{1, 2, \ldots, v\} : \; |b| = k\},$$

- edges:

$$b_1 \sim b_2 \Leftrightarrow |b_1 \cap b_2| = k - 1,$$

- $J(v, k) \cong J(v - k, k)$ (so that we will assume $k \leq v/2$).



# Completely regular codes in $J(v,k)$

### Theorem (P. Delsarte, 1973)

The vertices of a completely regular code in $J(v,k)$ are the blocks of a $t$-design with parameters $(v,k,\lambda)$ (for some $\lambda$).

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A *$t$-design* with parameters $(v,k,\lambda)$:
a collection $\mathcal{D}$ of some $k$-subsets (called *blocks*) of a set $X := \{1,2,\ldots,v\}$ such that every $t$ elements of $X$ are contained togheter in exactly $\lambda$ blocks.

The *strength* of $\mathcal{D}$: the largest $t$ such that $\mathcal{D}$ is a $t$-design.

# Strength of completely regular code in $J(v, k)$

The distinct eigenvalues of $J(v, k)$:

$$\theta_i = (k - i)(v - k - i) - i, \; 0 \leq i \leq k,$$

$$k(v - k) = \theta_0 > \theta_1 > \ldots > \theta_k = -k.$$

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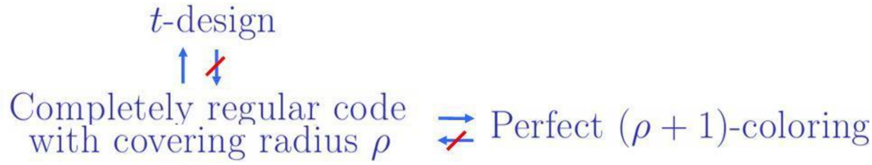
The strength of $C$ is the smallest $t$ such that $\theta_{t+1}$ is an eigenvalue of $P$ (note that $\theta_0$ always is an eigenvalue of $P$).

# Notions in $J(v, k)$



# Notions in $J(v, k)$

$t$-design

$\uparrow \quad \not\downarrow$

Completely regular code with covering radius $\rho$ $\;\substack{\longrightarrow \\ \not\longleftarrow}\;$ Perfect $(\rho + 1)$-coloring

Partial case:

Completely regular code with covering radius 1 $=$ Perfect 2-coloring

# Completely regular code $C$ in $J(v, k)$

- If $C$ has strength $0$ (i.e., $P$ has an eigenvalue $\theta_1$ of $J(v, k)$), then there is a subset $S \subset \{1, 2, \ldots, v\}$ s.t.

  either $C = \{b \in V : b \subseteq S\}$ or $C = \{b \in V : S \subseteq b\}$.

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- If $C$ has strength $k - 1$ (i.e., $P$ has an eigenvalue $\theta_k$), then its vertices are the blocks of a $(k-1)$-design, and, conversely, the blocks of any $(k-1)$-$(v, k, \lambda)$-design give a completely regular code in $J(v, k)$.

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- If $C$ has strength $i$, $0 < i < k-1$, — ???

  ($i = 1$, $\delta(C) > 1$, $\rho > 1$ in Martin'94)

# Completely regular codes with $\rho = 1$ in $J(v,k)$

- All the possible quotient matrices of perfect 2-colorings of $J(v,k)$ with $v \leq 8$ are listed
  (S. Avgustinovich, I. Mogilnykh'10)
- Partial results on non-existence of some perfect 2-colorings of $J(9,3)$, $J(11,3)$, $J(12,5)$, $J(13,4)$,
  (I. Mogilnykh'09)
- In particular, for $J(9,3)$ all the possible quotient matrices are listed except

$$\begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix},$$

which was left as an open case*.

* S. Avgustinovich, I. Mogilnykh // Diskret. Analiz i Issled. Oper., 2010.

# Perfect 2-coloring of $J(v, 3)$

Let $\{V_1, V_2\}$ be a perfect 2-coloring of $J(v, 3)$ (i.e., $C = V_i$ is a completely regular code with $\rho = 1$).

- If $C$ has strength 0 then we know $C$ (up to isomorphism) from the result by Meyerowitz.

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- if $C$ has strength $k-1=2$ then its vertices are the blocks of a 2-design with parameters $(v, 3, \lambda)$.
  For a 2-design with $k=3$, the following necessary conditions are known to be sufficient:

$$\lambda(v-1) \equiv 0 \pmod 2, \ \lambda v(v-1) \equiv 0 \pmod 6.$$

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- from the above we may assume that $C$ has strength 1 (i.e., the quotient matrix has eigenvalue $\theta_2$ of $J(v,3)$).

# Perfect 2-colorings of $J(v,3)$ with $\theta_2$

For $m > 4$, there are only 3 constructions of perfect 2-colorings of $J(2m,3)$ known, with the following quotient matrices:

$$\begin{pmatrix} 3(2m-5) & 6 \\ 4(m-2) & 2m-1 \end{pmatrix}, \begin{pmatrix} 3(m-3) & 3m \\ m-2 & 5m-7 \end{pmatrix},$$

(C. Godsil, C. Praeger; S. Avgustinovich, I. Mogilnykh'10)

$$\text{and } \begin{pmatrix} 3(m-1) & 3(m-2) \\ m+4 & 5m-13 \end{pmatrix}.$$

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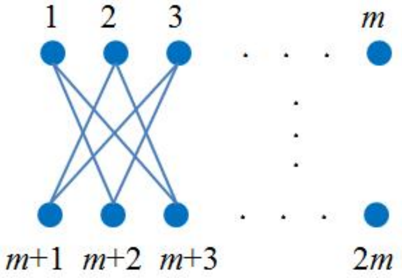
A perfect 2-coloring of $J(6,3)$ with quotient matrix:

$$\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}.$$

(S. Avgustinovich & I. Mogilnykh'08)

# Orbit coloring of $J(2m,3)$

Let $\Gamma$ be a $K_{m,m}$ without perfect matching ($i \not\sim i+m$) and $V(\Gamma) = \{1, 2, \ldots, 2m\}$.



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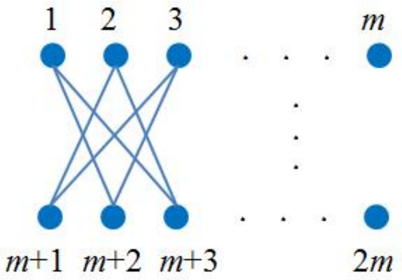


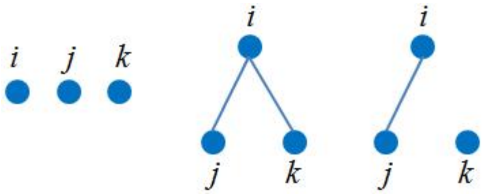

Let $G \cong \mathrm{Aut}(\Gamma)$. Consider the orbits of

$$\binom{V(\Gamma)}{3} := \{\{i, j, k\} | \ i, j, k \in V(\Gamma)\}$$

under the action of $G$: they correspond to the orbits of $V(J(2m, 3))$ under the action of $G \leq \mathrm{Aut}(J(2m, 3))$.

# Orbit coloring of $J(2m,3)$

The orbits of $\binom{V(\Gamma)}{3}$ under the action of $\mathrm{Aut}(\Gamma)$:




These give a perfect 3-coloring of $J(2m,3)$ with quotient matrix

$$P = \begin{pmatrix} 3(m-3) & 3(m-2) & 6 \\ m-2 & 5(m-3)+2 & 6 \\ m-2 & 3(m-2) & 2m-1 \end{pmatrix}.$$

Merging of a pair of these 3 orbits gives a perfect 2-coloring of $J(2m,3)$ with one of 3 quotient matrices above.

# Our results

*Theorem* 1

If $v$ is odd then $J(v, 3)$ does not contain a completely regular code with covering radius one and strength one.

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If $v$ is odd then $J(v,3)$ does not contain a completely regular code with covering radius one and strength one.

*Theorem* 2
If $v$ is even and $P$ is a *symmetric* quotient matrix of a completely regular code with covering radius one and strength one in $J(v,3)$ then

$$v = 6, \ P = \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$$

or

$$v = 10, \ P = \begin{pmatrix} 12 & 9 \\ 9 & 12 \end{pmatrix} = \begin{pmatrix} 3(5-1) & 3(5-2) \\ 5+4 & 5 \cdot 5 - 13 \end{pmatrix}.$$

# Conjecture

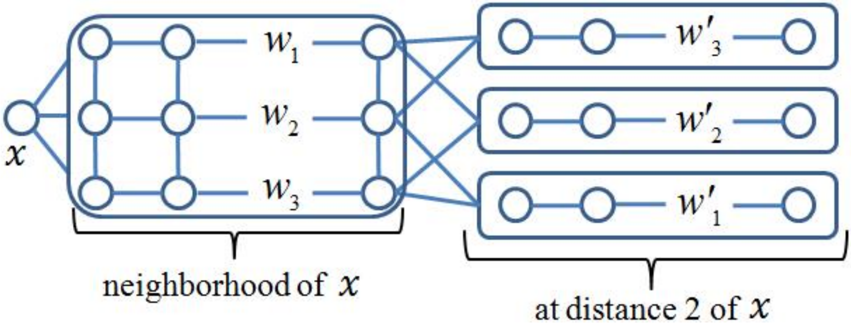
If $C$ is a completely regular code in $J(v,3)$, $v > 6$, with strength one and covering radius one, then $v = 2m$ and $C$ has one of the following quotient matrices:

$$\begin{pmatrix} 3(2m-5) & 6 \\ 4(m-2) & 2m-1 \end{pmatrix}, \begin{pmatrix} 3(m-3) & 3m \\ m-2 & 5m-7 \end{pmatrix},$$

$$\text{or} \begin{pmatrix} 3(m-1) & 3(m-2) \\ m+4 & 5m-13 \end{pmatrix}.$$

# Some arguments of the proof

Let $\{V_1, V_2\}$ be a perfect 2-coloring of $J(v, 3)$.
Call vertices from $V_1$ *white*, from $V_2$ — *black*.




$$w_i := \#\{\text{white vertices in } i\text{th row around } x\}$$

$$w_1 + w_2 + w_3 \sim \text{ color of } x$$

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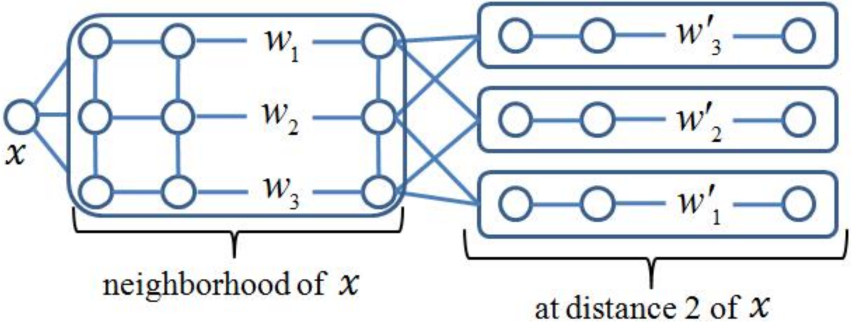


$$\begin{pmatrix} \text{rank } 3 \end{pmatrix} \begin{pmatrix} w_1' \\ w_2' \\ w_3' \end{pmatrix} = \begin{pmatrix} \sim w_i, \text{ colors of } x, \\ \text{and of vertices} \\ \text{in the column} \end{pmatrix}.$$

$w_i' \in \mathbf{N} \cup \{0\} \Rightarrow$ constraints on $w_i$ and colors in the column

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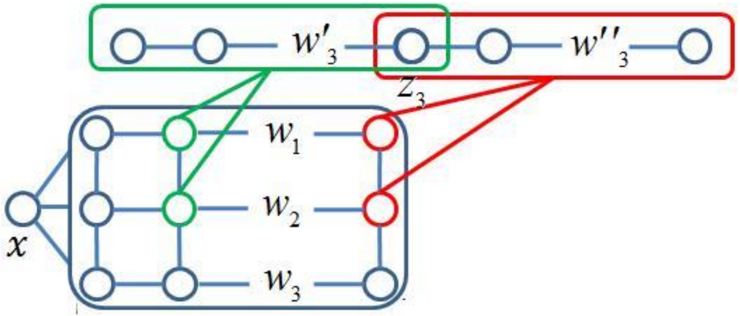


$$\begin{pmatrix} \text{rank } 2 \end{pmatrix} \begin{pmatrix} z_i' - z_j' \\ z_i' - z_k' \end{pmatrix} = \begin{pmatrix} \sim \ \{w_i\}, \ \{w_i'\}, \ \{w_i''\}, \\ \text{colors of } x \\ \text{of vertices in the columns} \end{pmatrix}$$

$\Rightarrow$ constraints on $\{w_i\}, \{w_i'\}, \{w_i''\}$ and the columns

# Example

Suppose that $P$ is symmetric and with $\theta_2$.

Then one can show that $P = \begin{pmatrix} 2v-8 & v-1 \\ v-1 & 2v-8 \end{pmatrix}$.

Suppose colors in two columns around a white vertex $x$.

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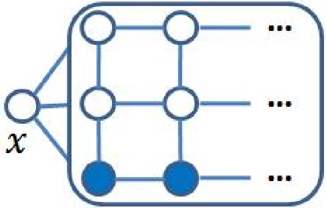

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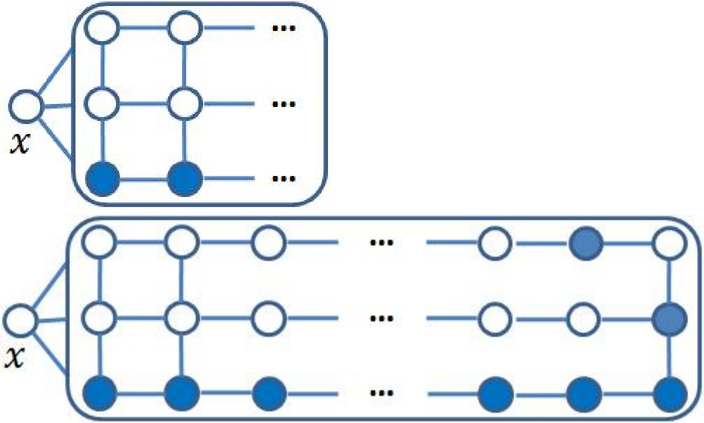

# Questions

- to classify all the realizable quotient matrices of perfect 2-colorings of $J(v,3)$,
- $J(v,4)$ is an open case (some constructions based on Steiner quadruple systems were recently found by S. Avgustinovich and I. Mogilnykh),
- to study codes in the Grassmann graphs $G_q(n,e)$ (our arguments work if $e=2$).