Perfect 2-colorings of Johnson graphs $J(v, 3)$

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Perfect coloring of a graph $\Gamma$ with $t$ colors

Equitable partition into $t$ parts $\equiv$ Perfect $t$-coloring

(just alternative name given by Dmitry Fon-Der-Flaass)
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- $V(\Gamma) = \dot{V}_1 \cup \dot{V}_2 \cup \ldots \cup \dot{V}_t$, ($V_i$ — the vertices of color $i$),
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- $V(\Gamma) = V_1 \dot{\cup} V_2 \dot{\cup} \ldots \dot{\cup} V_t$, ($V_i$ — the vertices of color $i$),
- every vertex of $V_i$ has exactly $p_{ij}$ neighbours of $V_j$, 
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- $V(\Gamma) = V_1 \cup V_2 \cup \ldots \cup V_t$, ($V_i$ — the vertices of color $i$),
- every vertex of $V_i$ has exactly $p_{ij}$ neighbours of $V_j$,
- $P := (p_{ij})_{t \times t}$ — the quotient matrix of coloring.

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$
Background

- classification of perfect error-correction codes over $\mathbb{F}_q$
  (V. Zinoviev & V. Leontiev’73; A. Tietäväinen’73)
- a notion of completely regular codes in association schemes
  (P. Delsarte’73)
- completely regular codes in Hamming and Johnson graphs/schemes
  (P. Delsarte’73)
  (A. Meyerowitz’03)
  (W. Martin’9x)
Distance partition w.r.t. a code

Let $C \subseteq V(\Gamma)$ be a code in $\Gamma$. For a vertex $x \in V(\Gamma)$, define

$$\text{dist}(x, C) := \min \{ \text{dist}(x, y) \mid y \in C \},$$

$$\rho_C := \max \{ \text{dist}(x, C) \mid x \in V(\Gamma) \}$$

– the covering radius of $C$. 
Distance partition w.r.t. a code

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the covering radius of $C$.

The distance partition of $V(\Gamma)$ with respect to $C$:

$$V(\Gamma) = \Gamma_0(C) \cup \Gamma_1(C) \cup \ldots \cup \Gamma_{\rho_C}(C),$$

where $\Gamma_i(C) := \{x \in V(\Gamma) : \text{dist}(x, C) = i\}$. 

![Diagram showing the distance partition of $V(\Gamma)$ with respect to a code $C$. The code $C$ is shaded and the distance classes $\Gamma_0(C)$, $\Gamma_1(C)$, ..., $\Gamma_{\rho_C}(C)$ are marked.]
Completely regular code

A code \( C \) is called \textit{completely regular} if the distance partition of \( V(\Gamma) \) w.r.t. \( C \) is a \textit{perfect} \((\rho_C + 1)\)-coloring:

\[ \forall \text{vertex } x \in V(\Gamma) \text{ of } C \text{ has } \gamma_i \text{ neighbours of } V(\Gamma)_{i-1}(C), \alpha_i \text{ of } V(\Gamma)_i(C), \beta_i \text{ of } V(\Gamma)_{i+1}(C), \]
Completely regular code

A code $C$ is called \emph{completely regular} if the distance partition of $V(\Gamma)$ w.r.t. $C$ is a perfect $(\rho_C + 1)$-coloring: there are integers $\alpha_i, \beta_i, \gamma_i$, $0 \leq i \leq \rho_C$, such that $\forall$ vertex $x \in \Gamma_i(C)$ has $\gamma_i$ neighbours of $\Gamma_{i-1}(C)$, $\alpha_i$ of $\Gamma_{i}(C)$, $\beta_i$ of $\Gamma_{i+1}(C)$

(A. Neumaier’92, P. Delsarte’73)
Completely regular code

The quotient matrix of this perfect coloring can be written in the following tridiagonal form:

\[ P = \begin{pmatrix}
    \alpha_0 & \beta_0 & 0 & \ldots & 0 \\
    \gamma_1 & \alpha_1 & \beta_1 \\
    0 & \gamma_2 & \alpha_2 & \beta_2 \\
    \vdots & \vdots & \vdots & \ddots & \ddots \\
    0 & \ldots & 0 & \gamma_\rho & \alpha_\rho
\end{pmatrix}. \]
Adjacency and quotient matrices

The adjacency matrix of a graph $\Gamma = (V, E)$:

$$(A)_{x,y} := \begin{cases} 
1 & \text{if } \{x, y\} \in E(\Gamma), \\
0 & \text{if } \{x, y\} \notin E(\Gamma).
\end{cases}$$

Eigenvalues:
Let a graph $\Gamma$ have a perfect $t$-coloring with quotient matrix $P$. Then every eigenvalue of $P$ is an eigenvalue of the adjacency matrix $A$ of $\Gamma$. 
Johnson graph $J(v, k)$

- **vertices:**
  \[ V = \{ b \subset \{1, 2, \ldots, v\} : |b| = k \}, \]

- **edges:**
  \[ b_1 \sim b_2 \iff |b_1 \cap b_2| = k - 1, \]

- $J(v, k) \cong J(v - k, k)$ (so that we will assume $k \leq v/2$).
Completely regular codes in $J(v, k)$

Theorem (P. Delsarte, 1973)

The vertices of a completely regular code in $J(v, k)$ are the blocks of a $t$-design with parameters $(v, k, \lambda)$ (for some $\lambda$).
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A **$t$-design** with parameters $(v, k, \lambda)$: a collection $\mathcal{D}$ of some $k$-subsets (called **blocks**) of a set $X := \{1, 2, \ldots, v\}$ such that every $t$ elements of $X$ are contained together in exactly $\lambda$ blocks.

The **strength** of $\mathcal{D}$: the largest $t$ such that $\mathcal{D}$ is a $t$-design.
Strength of completely regular code in $J(v, k)$

The distinct eigenvalues of $J(v, k)$:

$$\theta_i = (k - i)(v - k - i) - i, \ 0 \leq i \leq k,$$

$$k(v - k) = \theta_0 > \theta_1 > \ldots > \theta_k = -k.$$
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Let $C$ be a completely regular code in $J(v, k)$ with quotient matrix $P$. The eigenvalues of $P$ are some numbers of $\theta_0, \ldots, \theta_k$. 
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Let $C$ be a completely regular code in $J(v, k)$ with quotient matrix $P$. The eigenvalues of $P$ are some numbers of $\theta_0, \ldots, \theta_k$.

The strength of $C$ is the smallest $t$ such that $\theta_{t+1}$ is an eigenvalue of $P$ (note that $\theta_0$ always is an eigenvalue of $P$).
Notions in $J(v, k)$

$t$-design

\[ \uparrow \quad \downarrow \]

Completely regular code with covering radius $\rho$

\[ \rightarrow \]

Perfect $(\rho + 1)$-coloring
Notions in $J(v, k)$

$t$-design

\[ \uparrow \quad \downarrow \]

Completely regular code with covering radius $\rho$ \iff Perfect $(\rho + 1)$-coloring

Partial case:

Completely regular code with covering radius $1$ \iff Perfect $2$-coloring
Completely regular code $C$ in $J(v, k)$

- If $C$ has strength $0$ (i.e., $P$ has an eigenvalue $\theta_1$ of $J(v, k)$), then there is a subset $S \subset \{1, 2, \ldots, v\}$ s.t.

  either $C = \{b \in V : b \subseteq S\}$ or $C = \{b \in V : S \subseteq b\}$. 

  (A. Meyerowitz’03)
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- If $C$ has strength $k - 1$ (i.e., $P$ has an eigenvalue $\theta_k$), then its vertices are the blocks of a $(k - 1)$-design, and, conversely, the blocks of any $(k - 1)$-$(v, k, \lambda)$-design give a completely regular code in $J(v, k)$.

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Completely regular code $C$ in $J(v, k)$

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- If $C$ has strength $i$, $0 < i < k - 1$, ???

  ($i = 1$, $\delta(C) > 1$, $\rho > 1$ in Martin’94)
Completely regular codes with $\rho = 1$ in $J(v, k)$

- All the possible quotient matrices of perfect 2-colorings of $J(v, k)$ with $v \leq 8$ are listed
  (S. Avgustinovich, I. Mogilnykh’10)

- Partial results on non-existence of some perfect 2-colorings of $J(9, 3)$, $J(11, 3)$, $J(12, 5)$, $J(13, 4)$,
  (I. Mogilnykh’09)

- In particular, for $J(9, 3)$ all the possible quotient matrices are listed except

  \[
  \begin{pmatrix}
  10 & 8 \\
  8 & 10
  \end{pmatrix},
  \]

  which was left as an open case*. 

Perfect 2-coloring of $J(v, 3)$

Let $\{V_1, V_2\}$ be a perfect 2-coloring of $J(v, 3)$ (i.e., $C = V_i$ is a completely regular code with $\rho = 1$).

- If $C$ has strength 0 then we know $C$ (up to isomorphism) from the result by Meyerowitz.

(M. Dehon'61) from the above we may assume that $C$ has strength 1 (i.e., the quotient matrix has eigenvalue $\theta_2$ of $J(v, 3)$).
Perfect 2-coloring of $J(v, 3)$

Let $\{V_1, V_2\}$ be a perfect 2-coloring of $J(v, 3)$ (i.e., $C = V_i$ is a completely regular code with $\rho = 1$).

- If $C$ has strength 0 then we know $C$ (up to isomorphism) from the result by Meyerowitz.
- if $C$ has strength $k - 1 = 2$ then its vertices are the blocks of a 2-design with parameters $(v, 3, \lambda)$.

For a 2-design with $k = 3$, the following necessary conditions are known to be sufficient:

$$\lambda(v - 1) \equiv 0 \pmod{2}, \quad \lambda v(v - 1) \equiv 0 \pmod{6}.$$  

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Perfect 2-colorings of $J(v, 3)$ with $\theta_2$

For $m > 4$, there are only 3 constructions of perfect 2-colorings of $J(2m, 3)$ known, with the following quotient matrices:

$$
\begin{pmatrix}
3(2m - 5) & 6 \\
4(m - 2) & 2m - 1
\end{pmatrix},
\begin{pmatrix}
3(m - 3) & 3m \\
m - 2 & 5m - 7
\end{pmatrix},
$$

(C. Godsil, C. Praeger; S. Avgustinovich, I. Mogilnykh’10)

and

$$
\begin{pmatrix}
3(m - 1) & 3(m - 2) \\
m + 4 & 5m - 13
\end{pmatrix}.
$$

(S. Avgustinovich & I. Mogilnykh’10)
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\end{pmatrix}.
\]

(S. Avgustinovich & I. Mogilnykh’10)

A perfect 2-coloring of $J(6, 3)$ with quotient matrix:

\[
\begin{pmatrix}
4 & 5 \\
5 & 4
\end{pmatrix}.
\]

(S. Avgustinovich & I. Mogilnykh’08)
Orbit coloring of $J(2m, 3)$

Let $\Gamma$ be a $K_{m,m}$ without perfect matching ($i \not\sim i + m$) and $V(\Gamma) = \{1, 2, \ldots, 2m\}$. 

![Diagram of $K_{m,m}$ with orbit coloring]
Let $\Gamma$ be a $K_{m,m}$ without perfect matching ($i \not\sim i + m$) and $V(\Gamma) = \{1, 2, \ldots, 2m\}$.

Let $G \cong \text{Aut}(\Gamma)$. Consider the orbits of $\left( V(\Gamma) \right)_3 := \{\{i, j, k\} | i, j, k \in V(\Gamma)\}$ under the action of $G$: they correspond to the orbits of $V(J(2m, 3))$ under the action of $G \leq \text{Aut}(J(2m, 3))$. 
Orbit coloring of $J(2m, 3)$

The orbits of $\left( V(\Gamma) \right)_3$ under the action of $\text{Aut}(\Gamma)$:

These give a perfect 3-coloring of $J(2m, 3)$ with quotient matrix

$$P = \begin{pmatrix}
3(m - 3) & 3(m - 2) & 6 \\
 m - 2 & 5(m - 3) + 2 & 6 \\
m - 2 & 3(m - 2) & 2m - 1
\end{pmatrix}.$$ 

Merging of a pair of these 3 orbits gives a perfect 2-coloring of $J(2m, 3)$ with one of 3 quotient matrices above.
Our results

*Theorem 1*
If \( v \) is odd then \( J(v, 3) \) does not contain a completely regular code with covering radius one and strength one.
Our results

Theorem 1
If $v$ is odd then $J(v, 3)$ does not contain a completely regular code with covering radius one and strength one.

Theorem 2
If $v$ is even and $P$ is a symmetric quotient matrix of a completely regular code with covering radius one and strength one in $J(v, 3)$ then

$$v = 6, \quad P = \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$$

or

$$v = 10, \quad P = \begin{pmatrix} 12 & 9 \\ 9 & 12 \end{pmatrix} = \begin{pmatrix} 3(5 - 1) & 3(5 - 2) \\ 5 + 4 & 5 \cdot 5 - 13 \end{pmatrix}.$$
Conjecture

If $C$ is a completely regular code in $J(v, 3)$, $v > 6$, with strength one and covering radius one, then $v = 2m$ and $C$ has one of the following quotient matrices:

$$
\begin{pmatrix}
3(2m - 5) & 6 \\
4(m - 2) & 2m - 1
\end{pmatrix},
\begin{pmatrix}
3(m - 3) & 3m \\
m - 2 & 5m - 7
\end{pmatrix},
$$

or

$$
\begin{pmatrix}
3(m - 1) & 3(m - 2) \\
m + 4 & 5m - 13
\end{pmatrix}.
$$
Some arguments of the proof

Let \( \{V_1, V_2\} \) be a perfect 2-coloring of \( J(v, 3) \).
Call vertices from \( V_1 \) white, from \( V_2 \) — black.

\[ w_i := \#\{\text{white vertices in } i\text{th row around } x\} \]

\[ w_1 + w_2 + w_3 \sim \text{color of } x \]
Some arguments of the proof

\[
\begin{pmatrix}
\text{rank 3} \\
\end{pmatrix}
\begin{pmatrix}
w'_1 \\
w'_2 \\
w'_3
\end{pmatrix}
= 
\begin{pmatrix}
\sim w_i, \text{ colors of } x, \\
\text{and of vertices in the column}
\end{pmatrix}
\]

\[w'_i \in \mathbb{N} \cup \{0\} \Rightarrow \text{constraints on } w_i \text{ and colors in the column}\]
Some arguments of the proof

\[
\begin{pmatrix}
\text{rank 2} \\
Z_i' - Z_j' \\
Z_i' - Z_k'
\end{pmatrix}
= \begin{pmatrix}
\sim \{w_i\}, \{w_i'\}, \{w_i''\}, \\
\text{colors of } x \\
of \text{vertices in the columns}
\end{pmatrix}
\]

\[\Rightarrow \text{constraints on } \{w_i\}, \{w_i'\}, \{w_i''\} \text{ and the columns}\]
Example

Suppose that $P$ is symmetric and with $\theta_2$.
Then one can show that $P = \begin{pmatrix} 2v - 8 & v - 1 \\ v - 1 & 2v - 8 \end{pmatrix}$.
Suppose colors in two columns around a white vertex $x$. 
Example

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Example

Suppose that $P$ is symmetric and with $\theta_2$. Then one can show that $P = \begin{pmatrix} 2v - 8 & v - 1 \\ v - 1 & 2v - 8 \end{pmatrix}$.

Suppose colors in two columns around a white vertex $x$. 
Questions

- to classify all the realizable quotient matrices of perfect 2-colorings of $J(v, 3)$,
- $J(v, 4)$ is an open case (some constructions based on Steiner quadruple systems were recently found by S. Avgustinovich and I. Mogilnykh),
- to study codes in the Grassmann graphs $G_q(n, e)$ (our arguments work if $e = 2$).