On Q-polynomial Graphs of Type 2

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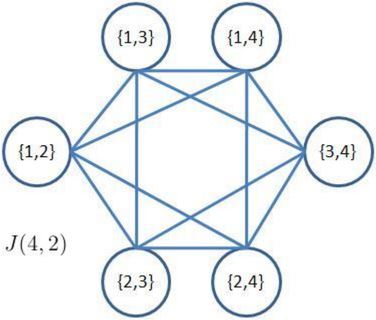
Notations

- All graphs in this talk are simple (no loops or multiple edges).
- We write $x \sim y$ if x and y are adjacent.
- $d(x, y) =$ the distance between x and y.
- $\Gamma_i(x) = \{y \mid d(x, y) = i\}$ so that $\Gamma_1(x) = \{y \mid x \sim y\}$.

The folded Johnson graph $\overline{J(2n,n)}$

- Let $X := \{1,2,\ldots,2n\}$ be a $2n$-element set.
- $\overline{J(2n,n)}$ has as vertices all pairs of complementary n-subsets $\{A,B\}$, i.e., $A \cup B = X$, $A \cap B = \emptyset$.
- $\{A,B\} \sim \{A',B'\}$ if and only if $|A \cap A'| \in \{1, n-1\}$.



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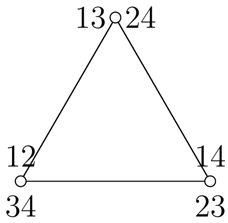
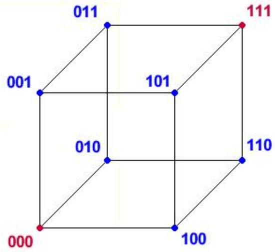


n-cube and its halved graph, $\frac{1}{2} - n$-cube

- n-cube has as vertices all words of length n over $\{0, 1\}$,
- $\mathbf{x} \sim \mathbf{y}$ if they differ in exactly one position.



$\frac{1}{2}$-n-cube (the halved graph of the n-cube) has the vertex set consisting of all even-weight words, with two words adjacent if they differ in exactly two positions.

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$\overline{\frac{1}{2} - n}$-cube:
If n is even, we can merge 'antipodal' vertices (which at maximal distance from each other) $\Rightarrow$ folded halved n-cube.

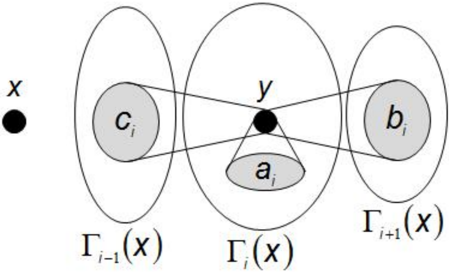
Distance-regular graphs (DRGs)

$\overline{J(4D+2, 2D+1)}$, $\overline{\frac{1}{2} - (4D+2)}$-cube, and $\frac{1}{2} - (2D+1)$-cube are distance-regular with intersection numbers:

$$c_i = \frac{hi(i-t+x)(i-t+y)(i-t+D)}{(2i-t)(2i-t-1)}, \ 1 \le i \le D,$$

$$b_i = \frac{h(i-t)(i-x)(i-y)(i-D)}{(2i-t)(2i-t+1)}, \ 0 \le i \le D-1$$

for some $h, x, y, t \in \mathbb{C}$.



Q-polynomial distance-regular graphs of type 2

A distance-regular graph Γ with diameter $D \geq 3$ is said to be Q-polynomial of type 2 if $\exists\ h, x, y, t, t^* \in \mathbb{C}$ such that the intersection numbers of Γ are given by:

$$c_i = \frac{hi(i-t+x)(i-t+y)(i-t+D)}{(2i-t)(2i-t-1)},\ 1 \leq i \leq D-1,$$

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In particular, the distinct eigenvalues of Γ are given by:

$$\theta_i = b_0 + hi(i-t^*),\ 0 \leq i \leq D,$$

where $t + t^* = x + y + D + 1$.

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Distance-regular graphs

All *known* primitive DRGs with unbounded diameter are Q-polynomial. The classification problem of Q-polynomial DRGs is one of main problems of the DRGs theory.

Leonard's Theorem says that the intersection numbers of Q-DRG may take one of seven possible forms:

- 1,
- $1A$ ($\nexists$, Terwilliger),
- 2,
- $2A$ (Neumaier, Terwilliger),
- $2B$ (Terwilliger),
- $2C$ (Egawa),
- 3 (Terwilliger).

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Classification of type 2 graphs

(Terwilliger, 1986)

A type 2 Q-DRG with diameter D at least 14 is either:

(i) the folded Johnson graph $\overline{J(4D+2, 2D+1)}$,

(ii) the $\frac{1}{2} - (2D+1)$-cube,

(iii) the folded halved cube, $\overline{\frac{1}{2} - (4D+2)}$-cube.

(iv) a graph not listed above, but with the same intersection array as (i) or (iii).

Many authors worked on characterization of $\overline{J(4D+2, 2D+1)}$ and the $\overline{\frac{1}{2} - (4D+2)}$-cube by intersection arrays. The problem was completely solved in

A.L. Gavrilyuk, J.H. Koolen, The Terwilliger polynomial of a Q-polynomial distance-regular graph and its application to the pseudo-partition graphs // arXiv:1403.4027.

Approach

- Terwilliger noticed that for all known examples of type 2 Q-DRGs the following holds:

$$c_3 - 3c_2 + 3 = b_2 - 2b_1 + b_0 - c_2 + 2 = 0 \qquad (*)$$

- If $(*)$ holds then the intersection numbers b_i, c_i take simple forms, which lead to known examples.
- So, we want to show that $(*)$ holds in general.
- In his work, it was shown that $t > D$ holds and

$$(1): \quad 5 < t < 2D - 1 \;\; \text{yields to} \;\; (*),$$

and

$$(2): \quad t > 27 \;\; \text{yields to} \;\; (*).$$

- Combining these facts, we see that if $D \geq 14$ then (1) or (2) is always true, so $(*)$ holds.

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$$(1): \quad 5 < t < 2D - 1 \;\text{ yields to }\; (*),$$

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$$(2): \quad t > 27 \;\text{ yields to }\; (*).$$

Our problem is a gap between $t < 2D - 1$ and $t > 27$.

$t > 27$

Lemma (Terwilliger)

Let Γ be a DRG with diameter D and at least one quad. Then for all $1 \leq i \leq D$

$$c_i - c_{i-1} + b_{i-1} - b_i - a_1 - 2 \geq 0 \qquad (**),$$

with equality at i if and only if for $\forall$ quad (q_1, q_2, q_3, q_4)

$$d(u, q_1) + d(u, q_3) = d(u, q_2) + d(u, q_4)$$

for $\forall u \in \Gamma$ with $\min\{d(u, q_j) \mid 1 \leq j \leq 4\} = i - 1.$

Equality at $i = 2 : \Leftrightarrow \Gamma$ does not contain:

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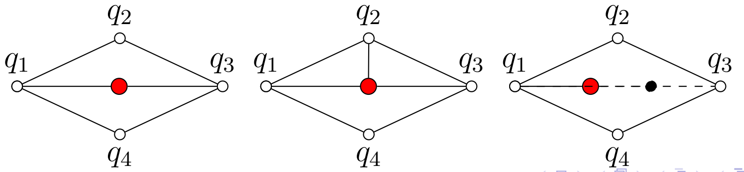

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Recall $(*)$:

$$c_3 - 3c_2 + 3 = b_2 - 2b_1 + b_0 - c_2 + 2 = 0$$

Inequality $(**)$ for $i = 2$:

$$c_2 - c_1 + b_1 - b_2 - a_1 - 2 = b_2 - 2b_1 + b_0 - c_2 + 2 = \text{RHS}(*) \geq 0$$

Inequality $(**)$ for $i = 3$:

$$c_3 - c_2 + b_2 - b_3 - a_1 - 2 = \alpha \text{LHS}(*) + \beta \text{RHS}(*) \geq 0$$

for some α, β.

Equalities in $(**)$ at $i = 2$ and $i = 3$ $\Leftrightarrow$ $\nexists$ some subgraphs

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Recall the distinct eigenvalues of Γ are given by:

$$\theta_i = b_0 + hi(i - t^*),\ 0 \leq i \leq D,$$

Consider $\mathbb{R}^{V(\Gamma)}$ and, for a vertex $x \in V(\Gamma)$, define $e_x \in \mathbb{R}^{V(\Gamma)}$ by $e_x = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 is in coordinate x.

Let E_1 be the matrix representing orthogonal projection onto the θ_1-eigenspace of the adjacency matrix A_1. Then $E_1^2 = E_1$ and

$$E_1 = \theta_0^* A_0 + \theta_1^* A_1 + \ldots + \theta_D^* A_D,$$

where A_i are the i-distance matrices of Γ, and θ_i^* (*dual eigenvalues*) are expressed in terms of h, x, y, t, D.

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For a set of vertices $\{x_1, x_2, \ldots, x_n\}$, we may consider the Gramm matrix of a set of vectors:

$$E_1 e_{x_1}, E_1 e_{x_2}, \ldots, E_1 e_{x_n}.$$

This matrix should be PSD and its determinant can be expressed in h, x, y, t, D, as

$$\langle E_1 e_{x_i}, E_1 e_{x_j} \rangle = e_{x_i} E_1^2 e_{x_j} = e_{x_i} E_1 e_{x_j} = \theta^*_{\mathrm{d}(x_i, x_j)}.$$

In fact, its determinant is a polynomial in t only.

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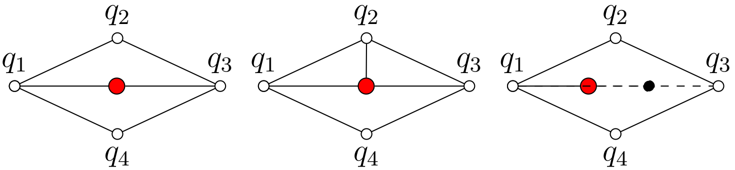
For these subgraphs:



the corresponding determinants are negative if $t > 27$.
This gives equality in $(**)$ for $i = 2$.

$$c_3 - 3c_2 + 3 = b_2 - 2b_1 + b_0 - c_2 + 2 = 0 \qquad (*)$$

$$c_i - c_{i-1} + b_{i-1} - b_i - a_1 - 2 \geq 0 \qquad (**)$$

Equalities in $(**)$ at $i = 2$ and $i = 3$ $\Leftrightarrow$ $\nexists$ some subgraphs

$$\Downarrow$$

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$t > 27$ – can we improve it?

Yes, we can.

— cannot appear already if $t > 8$.

But cannot appear if $t > 22.14$ only.

We can 'relax' such 'bad' subgraphs with additional vertices:

— cannot appear if $t > 14.23$.

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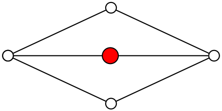

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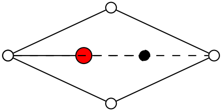
But  cannot appear if $t > 22.14$ only.

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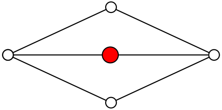

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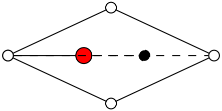
But  cannot appear if $t > 22.14$ only.

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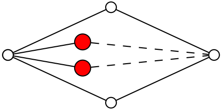

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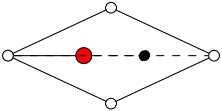
So, for example, we may admit that Γ contains



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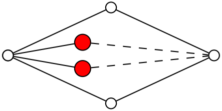

(if $D \geq 8$ then $t > 2D - 1 \geq 15 > 14.23$)

Using combinatorial arguments, we can show $c_2 \leq 6$ if $D \geq 8$.
In other words, if $c_2 > 6$ then $(*)$ holds $\Rightarrow \Gamma$ is known.

The Terwilliger polynomial

- Let Γ be a Q-polynomial DRG with diameter $D \geq 3$.
- Terwilliger (early 1990's): There exists a polynomial p_T of degree 4 such that, for any vertex $v \in \Gamma$, and any non-principal eigenvalue η of $\Gamma_1(v)$ we have $p_T(\eta) \geq 0$.
- p_T only depends on the intersection numbers and the Q-polynomial ordering of $E_0, E_1, \ldots, E_D$.
- We call p_T the **Terwilliger polynomial**.

The Terwilliger polynomial was described in '*Lecture Note on Terwilliger algebra*', the lectures given by P. Terwilliger in Japan and recorded by H. Suzuki.
See also:
A.L. Gavrilyuk, J.H. Koolen, The Terwilliger polynomial of a Q-polynomial distance-regular graph and its application to the pseudo-partition graphs // arXiv:1403.4027.

Result

Analysing the roots of this polynomial for type 2 graphs, we can show that

$$\text{the valency } b_0 \leq f(t, D, c_2).$$

Recall that we may assume $t < 27$, $D \leq 13$.

So, if $c_2 > 6$ and $D \geq 8$ then Γ is known.
If $c_2 \leq 6$ and $D \geq 8$ then we can bound the valency of Γ.

Theorem (G., Koolen, 2013-?)

A type 2 Q-DRG with diameter $D \geq 8$ is *known*, or $8 \leq D \leq 13$ and its valency b_0 is at most some number ($\simeq 6000$). (So, there may be only finitely many exceptions.)

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Thank you!