There exist no distance-regular graphs with intersection arrays $\{45, 30, 7; 1, 2, 27\}$ or $\{52, 35, 16; 1, 4, 28\}$

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Oisterwijk, 2011

Definition and notation

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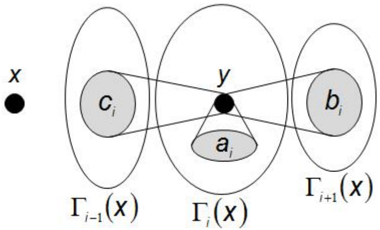
- Γ is called distance-regular (d.r.g. for short) if there are integers a_i, b_i, c_i, $0 \leq i \leq d := \mathrm{diam}(\Gamma)$, such that, for every pair of vertices $x, y \in \Gamma$ at distance i:

$$c_i = |\Gamma_1(y) \cap \Gamma_{i-1}(x)|,\ a_i = |\Gamma_1(y) \cap \Gamma_i(x)|,\ b_i = |\Gamma_1(y) \cap \Gamma_{i+1}(x)|.$$



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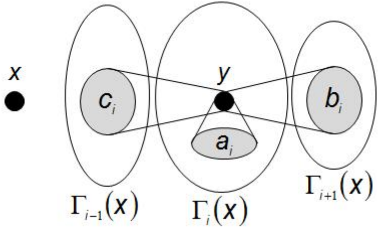

- $\{b_0, b_1, \ldots, b_{d-1}; c_1 = 1, c_2, \ldots, c_d\}$

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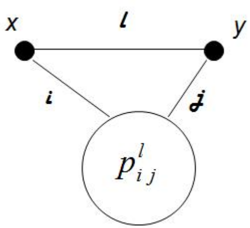
- $p^l_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$, where $\mathrm{dist}(x, y) = l$.



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A distance-regular Terwilliger graph Γ is a non-complete d.r.g. such that, for every pair of vertices $x, y \in \Gamma$ at distance 2, the subgraph $\Gamma_1(x) \cap \Gamma_1(y)$ is complete.

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There are only 3 d.r. Terwilliger graphs known with $c_2 \geq 2$:

- the icosahedron with intersection array $\{5, 2, 1; 1, 2, 5\}$;
- the Doro graph with intersection array $\{10, 6, 4; 1, 2, 5\}$;
- the Conway–Smith graph with intersection array $\{10, 6, 4, 1; 1, 2, 6, 10\}$.

s-claw inequality (J.H. Koolen, J.Park, 2010)

[1, Lemma 2 and Proposition 3]

- If, for a vertex $x \in \Gamma$, the subgraph $\Gamma_1(x)$ contains an s-coclique, $s \geq 2$, then

$$c_2 - 1 \geq \frac{s(a_1 + 1) - b_0}{\binom{s}{2}}.$$

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- Suppose s is a maximal number such that $\forall x \in \Gamma$, $\forall y, z \in \Gamma_1(x) : y \not\sim z$ there exists an s-coclique of $\Gamma_1(x)$, containing vertices y, z, then

$$c_2 - 1 \geq \max\{\frac{s'(a_1 + 1) - b_0}{\binom{s'}{2}} | 2 \leq s' \leq s\} \qquad (1)$$

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and equality implies that Γ is a Terwilliger graph.

s-claw inequality: application

[2, Theorem 4.2]

If $c_2 \geq 2$ then equality in (1) implies that $c_2 = 2$ and Γ is the icosahedron, the Doro graph, or the Conway–Smith graph.

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[1, Corollary 5]

There are no distance-regular graphs with one of the following intersection arrays: $\{44, 30, 5; 1, 3, 40\}$ and $\{65, 44, 11; 1, 4, 55\}$.

Some suspicious arrays

- $\{55, 36, 11; 1, 4, 45\}$, $s \leq 3$,
- $\{56, 36, 9; 1, 3, 48\}$, $s \leq 3$,
- $\{45, 30, 7; 1, 2, 27\}$, $s \leq 3$,
- $\{52, 35, 16; 1, 4, 28\}$, $s \leq 4$.

Some suspicious arrays

- $\{55, 36, 11; 1, 4, 45\}$ was ruled out in [3,4],
- $\{56, 36, 9; 1, 3, 48\}$ was ruled out in [3,4],
- $\{45, 30, 7; 1, 2, 27\}$,
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Geometric distance-regular graphs

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The Hoffman-Delsarte bound [5, Proposition 4.4.6(i)]

Let Γ be a d.r.g. with valency k and smallest eigenvalue θ_d. Then for a clique L of Γ:

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- A Delsarte clique: that contains $1 - k/\theta_d$ vertices exactly.
- A geometric d.r.g.: if there exists a set $\mathcal{L}$ of Delsarte cliques such that each edge of Γ lies in a unique clique $L \in \mathcal{L}$.
- Examples: a Hamming graph, a Johnson graph, a Grassmann graph..

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Theorem (S. Bang, 2011),[3]

Let Γ be a a geometric d.r.g. with smallest eigenvalue -3. Then Γ satisfies one of the following (12 cases).

- ...
- $(viii)$ Γ has diameter 3 and intersection array $\{3\alpha + 3, 2\alpha + 2, \alpha + 2 - \beta; 1, 2, 3\beta\}$, where $\alpha \geq \beta \geq 1$.
- ...

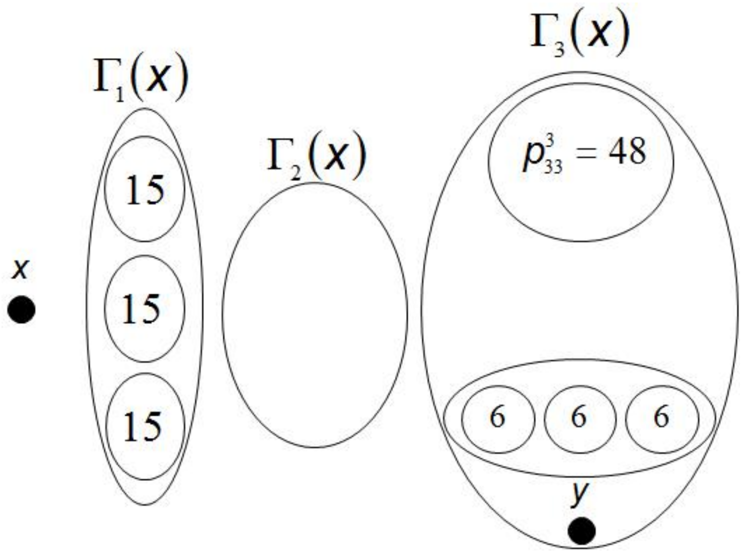
Geometric distance-regular graphs

In the list of [5] there are only the intersection arrays:

- of the Hamming graph $H(3, \alpha + 2)$ (where $\beta = 1$),
- of the Doob graph with diameter 3 (where $\beta = 1$),
- and the intersection array $\{45, 30, 7; 1, 2, 27\}$ (where $\alpha = 14, \beta = 9$)

that satisfy Case $(viii)$.

Properties of $\{45, 30, 7; 1, 2, 27\}$



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Lemma

For every pair of vertices $x, y \in \Gamma$ at distance 3, there is a 7-clique $L \subset \Gamma_3(x)$ such that $|L \cap \Gamma_3(x) \cap \Gamma_3(y)| > 4$.

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Proof:

- Assume the converse. Choose a 7-clique L of $\Gamma_3(x)$.
- the set $\Gamma_3(x) \setminus (\cup_{z \in L} \Gamma_1(z))$ contains 84 vertices.

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- Denote by N the number of pairs (z, y), where $z \in L$, $y \in \Gamma_3(x) \cap \Gamma_3(z)$;

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- Denote by N the number of pairs (z, y), where $z \in L$, $y \in \Gamma_3(x) \cap \Gamma_3(z)$; then

$$|L| \cdot p^3_{33} = 7 \cdot 48 \leq N \leq 84 \cdot 4 = 7 \cdot 48,$$

- this implies that $|L \cap \Gamma_3(y)| = 4$.

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$$|L| \cdot p^3_{33} = 7 \cdot 48 \leq N \leq 84 \cdot 4 = 7 \cdot 48,$$

- this implies that $|L \cap \Gamma_3(y)| = 4$.
- it is easily seen that $\Gamma_3(x)$ should be a d.r.g. with array $\{18, 12, 4; 1, 2, 9\}$ $\Rightarrow$ non-integer eigenvalue multiplicities!

Result

On the other hand, we can show that each of the cases $|L \cap \Gamma_3(x) \cap \Gamma_3(y)| \in \{5, 6, 7\}$ leads to a contradiction.

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Theorem [7]

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Problems

Problem 1

Are there distance-regular graphs with intersection array $\{3\alpha+3, 2\alpha+2, \alpha+2-\beta; 1, 2, 3\beta\}$, where $\alpha > \beta > 1$?

Problem 2 ($\Leftarrow$ Problem 1)

Are there the integers α, β such that $\alpha > 14$, $\alpha > \beta > 1$, and

$$\mathrm{mult}(-3) = \frac{1 + 3(\alpha+1) + 3(\alpha+1)^2 + \frac{(\alpha-\beta+2)(\alpha+1)^2}{\beta}}{1 + 3/(\alpha+1) + 3/(\alpha+1)^2 + \frac{\beta}{(\alpha-\beta+2)(\alpha+1)^2}}$$

— is integer, and β divides $(\alpha+1)^2(\alpha+2)$ and $3(\alpha+1)^2$?

We checked all the pairs α, β, where $\alpha < 10^4$.

An inequality concerning cocliques of μ-graphs

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$$c_2 - 2 \geq \frac{s(a_1 + 1) - (b_0 - b_2)}{\binom{s}{2}}.$$

Properties of a graph

Let Γ be a d.r.g. with intersection array $\{52, 35, 16; 1, 4, 28\}$ (Γ has a spectrum $52^1, 20^{65}, 4^{182}, -4^{520}$), and x, y be two its vertices at distance 2.

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Lemma

- a clique of Γ contains at most 13 vertices (by the Hoffman–Delsarte bound),
- $\Gamma_1(x)$ doesn't contain a 5-coclique (by the s-claw inequality),
- $\Gamma_1(x) \cap \Gamma_1(y)$ doesn't contain a 3-coclique (by the previous lemma).

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But if $\Gamma_1(x) \cap \Gamma_1(y)$ contains a 2-coclique, then we can show that Γ contains a 14-clique, which is impossible.

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Hence, Γ is a Terwilliger graph and this is impossible too (by [5, Corollary 1.16.6]).

Theorem

There is no distance-regular graph with intersection array $\{52, 35, 16; 1, 4, 28\}$.

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