There exist no distance-regular graphs with intersection arrays \(\{45, 30, 7; 1, 2, 27\}\) or \(\{52, 35, 16; 1, 4, 28\}\)

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Definition and notation

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- $\Gamma$ is called distance-regular (d.r.g. for short) if there are integers $a_i, b_i, c_i$, $0 \leq i \leq d := \text{diam}(\Gamma)$, such that, for every pair of vertices $x, y \in \Gamma$ at distance $i$:
  
  $$c_i = |\Gamma_1(y) \cap \Gamma_{i-1}(x)|, \quad a_i = |\Gamma_1(y) \cap \Gamma_i(x)|, \quad b_i = |\Gamma_1(y) \cap \Gamma_{i+1}(x)|.$$

![Diagram showing $\Gamma_i(x)$, $\Gamma_{i-1}(x)$, and $\Gamma_{i+1}(x)$ with $c_i$, $a_i$, and $b_i$ labels.]
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\begin{itemize}
    \item $\{b_0, b_1, \ldots, b_{d-1}; c_1 = 1, c_2, \ldots, c_d\}$
\end{itemize}
Definition and notation

- \( p_{ij}^l := |\Gamma_i(x) \cap \Gamma_j(y)| \), where \( \text{dist}(x, y) = l \).
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A distance-regular Terwilliger graph $\Gamma$ is a non-complete d.r.g. such that, for every pair of vertices $x, y \in \Gamma$ at distance 2, the subgraph $\Gamma_1(x) \cap \Gamma_1(y)$ is complete.
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There are only 3 d.r. Terwilliger graphs known with $c_2 \geq 2$:
- the icosahedron with intersection array $\{5, 2, 1; 1, 2, 5\}$;
- the Doro graph with intersection array $\{10, 6, 4; 1, 2, 5\}$;
- the Conway–Smith graph with intersection array $\{10, 6, 4, 1; 1, 2, 6, 10\}$. 
\textbf{s-claw inequality (J.H. Koolen, J.Park, 2010)}

\[ [1, \text{Lemma 2 and Proposition 3}] \]

- If, for a vertex \( x \in \Gamma \), the subgraph \( \Gamma_1(x) \) contains an \( s \)-coclique, \( s \geq 2 \), then

\[
c_2 - 1 \geq \frac{s(a_1 + 1) - b_0}{\binom{s}{2}}.
\]
**s-claw inequality (J.H. Koolen, J.Park, 2010)**

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- Suppose $s$ is a maximal number such that $\forall x \in \Gamma$, $\forall y, z \in \Gamma_1(x) : y \not\sim z$ there exists an $s$-coclique of $\Gamma_1(x)$, containing vertices $y, z$, then

  $$c_2 - 1 \geq \max\left\{ \frac{s'(a_1 + 1) - b_0}{\binom{s'}{2}} \middle| 2 \leq s' \leq s \right\} \quad (1)$$
s-claw inequality (J.H. Koolen, J. Park, 2010)

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  \tag{1}
\]

and equality implies that $\Gamma$ is a Terwilliger graph.
s-claw inequality: application

[2, Theorem 4.2]
If $c_2 \geq 2$ then equality in (1) implies that $c_2 = 2$ and $\Gamma$ is the icosahedron, the Doro graph, or the Conway–Smith graph.
\textbf{s-claw inequality: application}

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We may assume $s \geq \lceil \frac{b_0}{a_1+1} \rceil$. 
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[1, Corollary 5]
There are no distance-regular graphs with one of the following intersection arrays: \( \{44, 30, 5; 1, 3, 40\} \) and \( \{65, 44, 11; 1, 4, 55\} \).
Some suspicious arrays

- \{55, 36, 11; 1, 4, 45\}, \(s \leq 3\),
- \{56, 36, 9; 1, 3, 48\}, \(s \leq 3\),
- \{45, 30, 7; 1, 2, 27\}, \(s \leq 3\),
- \{52, 35, 16; 1, 4, 28\}, \(s \leq 4\).
Some suspicious arrays

- \{55, 36, 11; 1, 4, 45\} was ruled out in [3,4],
- \{56, 36, 9; 1, 3, 48\} was ruled out in [3,4],
- \{45, 30, 7; 1, 2, 27\},
- \{52, 35, 16; 1, 4, 28\}. 
Geometric distance-regular graphs

The array \( \{45, 30, 7; 1, 2, 27\} \) independently appeared in [3].
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**The Hoffman-Delsarte bound** [5, Proposition 4.4.6(i)]

Let \(\Gamma\) be a d.r.g. with valency \(k\) and smallest eigenvalue \(\theta_d\). Then for a clique \(L\) of \(\Gamma\):

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|L| \leq 1 - k/\theta_d.
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The Hoffman-Delsarte bound [5, Proposition 4.4.6(i)]

Let $\Gamma$ be a d.r.g. with valency $k$ and smallest eigenvalue $\theta_d$. Then for a clique $L$ of $\Gamma$:

$$|L| \leq 1 - k/\theta_d.$$ 

- A Delsarte clique: that contains $1 - k/\theta_d$ vertices exactly.
- A geometric d.r.g.: if there exists a set $\mathcal{L}$ of Delsarte cliques such that each edge of $\Gamma$ lies in a unique clique $L \in \mathcal{L}$.
- Examples: a Hamming graph, a Johnson graph, a Grassmann graph.
Geometric distance-regular graphs

Classification of geometric d.r.g. with smallest eigenvalue $-m$

- $m = 2$ (follows by A. Blokhuis, A. E. Brouwer, [6]),
Geometric distance-regular graphs

Classification of geometric d.r.g. with smallest eigenvalue \(-m\)

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- \(m = 3\) (S. Bang, [3]).
Geometric distance-regular graphs

Classification of geometric d.r.g. with smallest eigenvalue $-m$

- $m = 2$ (follows by A. Blokhuis, A. E. Brouwer, [6]),
- $m = 3$ (S. Bang, [3]).

Theorem (S. Bang, 2011),[3]

Let $\Gamma$ be a geometric d.r.g. with smallest eigenvalue $-3$. Then $\Gamma$ satisfies one of the following (12 cases).

- ...
- (viii) $\Gamma$ has diameter 3 and intersection array $\{3\alpha + 3, 2\alpha + 2, \alpha + 2 - \beta; 1, 2, 3\beta\}$, where $\alpha \geq \beta \geq 1$.
- ...

Geometric distance-regular graphs

In the list of [5] there are only the intersection arrays:

- of the Hamming graph $H(3, \alpha + 2)$ (where $\beta = 1$),
- of the Doob graph with diameter 3 (where $\beta = 1$),
- and the intersection array $\{45, 30, 7; 1, 2, 27\}$ (where $\alpha = 14, \beta = 9$)

that satisfy Case (viii).
Properties of \{45, 30, 7; 1, 2, 27\}
Lemma

For every pair of vertices $x, y \in \Gamma$ at distance 3, there is a 7-clique $L \subset \Gamma_3(x)$ such that $|L \cap \Gamma_3(x) \cap \Gamma_3(y)| > 4$.

Proof:
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Proof:

- Assume the converse. Choose a 7-clique \( L \) of \( \Gamma_3(x) \).
- The set \( \Gamma_3(x) \setminus (\cup_{z \in L} \Gamma_1(z)) \) contains 84 vertices.
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- $\forall y \in \Gamma_3(x) \setminus (\bigcup_{z \in L} \Gamma_1(z))$, we have $|L \cap \Gamma_3(x) \cap \Gamma_3(y)| \leq 4$.
- Denote by $N$ the number of pairs $(z, y)$, where $z \in L$, $y \in \Gamma_3(x) \cap \Gamma_3(z)$;
Properties of \{45, 30, 7; 1, 2, 27\}

Lemma

For every pair of vertices \(x, y \in \Gamma\) at distance 3, there is a 7-clique \(L \subset \Gamma_3(x)\) such that \(|L \cap \Gamma_3(x) \cap \Gamma_3(y)| > 4\).

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- Assume the converse. Choose a 7-clique \(L\) of \(\Gamma_3(x)\).
- The set \(\Gamma_3(x) \setminus (\bigcup_{z \in L} \Gamma_1(z))\) contains 84 vertices.
- \(\forall y \in \Gamma_3(x) \setminus (\bigcup_{z \in L} \Gamma_1(z))\), we have \(|L \cap \Gamma_3(x) \cap \Gamma_3(y)| \leq 4\).
- Denote by \(N\) the number of pairs \((z, y)\), where \(z \in L\), \(y \in \Gamma_3(x) \cap \Gamma_3(z)\); then

\[
|L| \cdot p_{33}^3 = 7 \cdot 48 \leq N \leq 84 \cdot 4 = 7 \cdot 48,
\]

- this implies that \(|L \cap \Gamma_3(y)| = 4\).
Lemma

For every pair of vertices $x, y \in \Gamma$ at distance 3, there is a 7-clique $L \subset \Gamma_3(x)$ such that $|L \cap \Gamma_3(x) \cap \Gamma_3(y)| > 4$.

Proof:

- Assume the converse. Choose a 7-clique $L$ of $\Gamma_3(x)$.
- the set $\Gamma_3(x) \setminus (\bigcup_{z \in L} \Gamma_1(z))$ contains 84 vertices.
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- Denote by $N$ the number of pairs $(z, y)$, where $z \in L$, $y \in \Gamma_3(x) \cap \Gamma_3(z)$; then

  $$|L| \cdot p_{33}^3 = 7 \cdot 48 \leq N \leq 84 \cdot 4 = 7 \cdot 48,$$

  this implies that $|L \cap \Gamma_3(y)| = 4$.

- it is easily seen that $\Gamma_3(x)$ should be a d.r.g. with array

  $\{18, 12, 4; 1, 2, 9\} \Rightarrow$ non-integer eigenvalue multiplicities!
On the other hand, we can show that each of the cases $|L \cap \Gamma_3(x) \cap \Gamma_3(y)| \in \{5, 6, 7\}$ leads to a contradiction.
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**Theorem [7]**

There is no distance-regular graph with intersection array \( \{45, 30, 7; 1, 2, 27\} \).
Problems

Problem 1
Are there distance-regular graphs with intersection array 
\( \{3\alpha + 3, 2\alpha + 2, \alpha + 2 - \beta; 1, 2, 3\beta\} \), where \( \alpha > \beta > 1 \)?

Problem 2 (\( \iff \) Problem 1)
Are there the integers \( \alpha, \beta \) such that \( \alpha > 14, \alpha > \beta > 1 \), and
\[
\text{mult}(-3) = \frac{1 + 3(\alpha + 1) + 3(\alpha + 1)^2 + (\alpha-\beta+2)(\alpha+1)^2}{1 + 3/(\alpha + 1) + 3/(\alpha + 1)^2 + \beta/(\alpha-\beta+2)(\alpha+1)^2}
\]
— is integer, and \( \beta \) divides \((\alpha + 1)^2(\alpha + 2)\) and \(3(\alpha + 1)^2\)?

We checked all the pairs \( \alpha, \beta \), where \( \alpha < 10^4 \).
An inequality concerning cocliques of $\mu$-graphs

Let $\Gamma$ be a distance-regular graph with diameter at least 3.
An inequality concerning cocliques of \(\mu\)-graphs

Let \(\Gamma\) be a distance-regular graph with diameter at least 3.

**Lemma**

If, for vertices \(x, y \in \Gamma\) at distance 2, the subgraph \(\Gamma_1(x) \cap \Gamma_1(y)\) contains an \(s\)-coclque, \(s \geq 2\), then

\[
c_2 - 2 \geq \frac{s(a_1 + 1) - (b_0 - b_2)}{\binom{s}{2}}.
\]
Properties of a graph

Let $\Gamma$ be a d.r.g. with intersection array $\{52, 35, 16; 1, 4, 28\}$ ($\Gamma$ has a spectrum $52^1, 20^{65}, 4^{182}, -4^{520}$), and $x, y$ be two its vertices at distance 2.
Properties of a graph

Let $\Gamma$ be a d.r.g. with intersection array $\{52, 35, 16; 1, 4, 28\}$ ($\Gamma$ has a spectrum $52^1, 20^{65}, 4^{182}, -4^{520}$), and $x, y$ be two its vertices at distance 2.

**Lemma**

- a clique of $\Gamma$ contains at most 13 vertices (by the Hoffman–Delsarte bound),
- $\Gamma_1(x)$ doesn’t contain a 5-coclique (by the $s$-claw inequality),
- $\Gamma_1(x) \cap \Gamma_1(y)$ doesn’t contain a 3-coclique (by the previous lemma).
The result

But if $\Gamma_1(x) \cap \Gamma_1(y)$ contains a 2-coclique, then we can show that $\Gamma$ contains a 14-clique, which is impossible.
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Hence, $\Gamma$ is a Terwilliger graph and this is impossible too (by [5, Corollary 1.16.6]).

**Theorem**

There is no distance-regular graph with intersection array
\[\{52, 35, 16; 1, 4, 28\}.\]
References

[4] A.L. Gavrilyuk: Distance-regular graphs with intersection arrays \{55, 36, 11; 1, 4, 45\}, \{56, 36, 9; 1, 3, 48\} do not exist.