On Characterization of Bilinear Forms Graphs

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Definitions

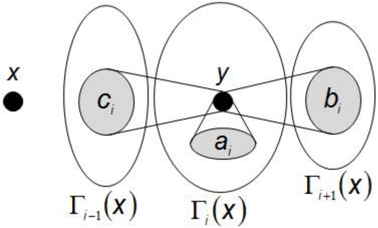
- All graphs in this talk are simple (no loops or multiple edges).
- We write $x \sim y$ if x and y are adjacent.
- $d(x, y)$ — the distance between x and y (the length of a shortest path between x and y).
- $D(\Gamma)$ — diameter of a graph Γ (max distance in Γ).

Distance-regular graphs

Define $\Gamma_i(x) = \{y \mid d(x,y) = i\}$ so that $\Gamma_1(x) = \{y \mid x \sim y\}$.

A connected graph Γ is called **distance-regular** (DRG):
if there are numbers a_i, b_i, c_i, $0 \le i \le D(\Gamma)$, s.t.
for $\forall$ pair of vertices x, y with $d(x,y) = i$

$|\Gamma_1(y) \cap \Gamma_{i-1}(x)| = c_i, \ \ |\Gamma_1(y) \cap \Gamma_i(x)| = a_i, \ \ |\Gamma_1(y) \cap \Gamma_{i+1}(x)| = b_i$



Γ is **regular** with valency $b_0 = |\Gamma_1(x)|$, $\forall x \in V(\Gamma)$ so that

$$b_0 = c_i + a_i + b_i.$$

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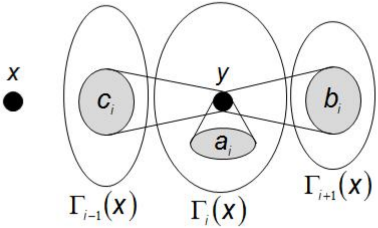

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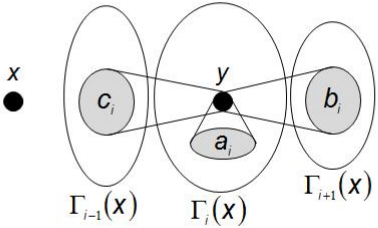

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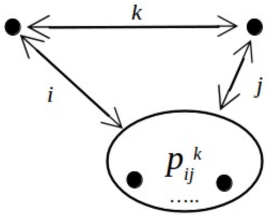
Distance-regular graphs

Let x, y be any pair of vertices of Γ with $d(x, y) = k$.
Then the **intersection numbers** of Γ

$$p^k_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)| = |\Gamma_j(x) \cap \Gamma_i(y)|$$

do not depend on the choice of x, y, $\forall\ 0 \le i, j, k \le D(\Gamma)$.



Note that $c_i = p^i_{1,i-1}$, $a_i = p^i_{1,i}$, $b_i = p^i_{1,i+1}$.

Distance-regular graphs

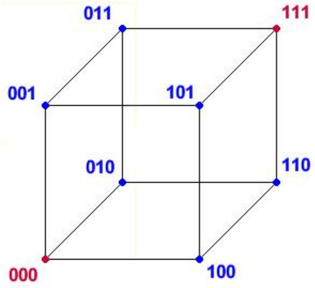
For a distance-regular graph Γ with diameter D, define its **intersection array**:

$$\iota(\Gamma) := \{b_0, b_1, \ldots, b_{D-1} \; ; \; c_1, c_2, \ldots, c_D\}.$$

All intersection numbers p_{ij}^k can be calculated from $\iota(\Gamma)$.

Examples: Hamming graph $H(n, q)$

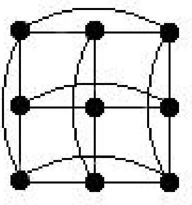
- Let $q \geq 2$, $n \geq 1$ be integers.
- $Q := \{1, 2, \ldots, q\}$.
- $H(n, q)$ has vertex set $Q^n = Q \times Q \times \ldots \times Q$.
- $\mathbf{x} \sim \mathbf{y}$ if they differ in exactly one position.
- Diameter equals n.



$$\Gamma = H(3, 2), \ \iota(\Gamma) = \{3, 2, 1; 1, 2, 3\}$$

Examples: $H(2, q)$, the lattice graph

Two vertices are adjacent iff they are in the same row or column.

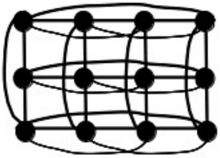


$$\Gamma = H(2,3),\ \iota(\Gamma) = \{2(q-1), q-1; 1, 2\}$$

The lattice graph

A complete graph: all vertices are pairwise adjacent.

$n \times m$-**lattice**: the Cartesian product of two complete graphs on n and m vertices.



Example: 3×4-lattice $\cong$ 4×3-lattice.

Examples: Grassmann graph $J_q(n, d)$

- Let $q \geq 2$ be a prime power, $n \geq d \geq 1$ be integers.
- $J_q(n, d)$ has as vertices all d-dim. subspaces $U \leq \mathbb{F}_q^n$.
- $U_1 \sim U_2$ if $dim(U_1 \cap U_2) = d - 1$.
- Diameter equals $\min(d, n - d)$.

Examples: Bilinear forms graphs $Bil_q(n \times m)$

- Let $n, m \geq 1$ be integers, q a prime power.
- $Bil_q(n \times m)$ has as vertices all $n \times m$-matrices over $\mathbb{F}_q$.
- $A \sim B$ if $\text{rank}(A - B) = 1$.
- $Bil_q(n \times m) \cong Bil_q(m \times n)$.
- Diameter equals $\min(n, m)$.
- Can be viewed as a subgraph of $J_q(n + m, m)$.

Open problems

We can construct $H(n,q)$, $J_q(n,d)$, $Bil_q(n,m)$ with any given diameter — there are quite few known constructions of DRGs with this property.

Moreover, all the primitive DRGs obtained from these constructions are *Q-polynomial* (I'll define this later).

Bannai's problem (early 1980's)

Can we classify the Q-polynomial distance-regular graphs with large diameter?

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Characterization by $\iota(\Gamma)$

$$\iota(\Gamma) := \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}.$$

Q-polinomiality can be recognized from $\iota(\Gamma)$.

Thus, one of steps towards solution of Bannai's problem is to characterize the known DRGs by their intersection arrays (i.e., to find all DRGs with given $\iota(\Gamma)$).

Theorem (Egawa, 1981)

Any DRG with the same intersection array as a Hamming graph is the Hamming graph $H(n, q)$ or, if $q = 4$, a Doob graph.

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Theorem (Metsch, 1995)

The Grassmann graph $J_q(n, d)$ is characterized by its intersection array if:

- $n \neq 2d$, $n \neq 2d \pm 1$,
- $n \neq 2d \pm 2$ if $q \in \{2, 3\}$,
- $n \neq 2d \pm 3$ if $q = 2$.

An exception, the *twisted Grassmann graph* with parameters as of $J_q(2d \pm 1, d)$ for all q, was constructed by E. van Dam and J.H. Koolen (2004).

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Theorem (Metsch, 1999)

The bilinear forms graph $Bil_q(n \times m)$ is characterized by its intersection array if:

- $q = 2$ and $m \geq n + 4$,
- $q \geq 3$ and $m \geq n + 3$.

Open cases:

- $q = 2$ and $m \in \{\not{n}, n+1, n+2, n+3\}$,
- $q \geq 3$ and $m \in \{n, n+1, n+2\}$.

Theorem (G., Koolen, 2012-2013)

The bilinear forms graph $Bil_q(n \times m)$ is characterized by its intersection array when $q = 2$ and $n = m$.

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Local structure

$\Gamma_1(x)$ = the **local** graph at vertex x.

The local graphs of:

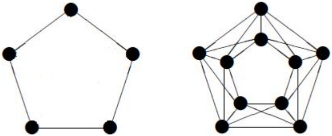
$J_q(n,d)$:

$$q\text{-clique extension of } \begin{bmatrix} n-d \\ 1 \end{bmatrix} \times \begin{bmatrix} d \\ 1 \end{bmatrix}\text{-lattice},$$

$Bil_q(n \times m)$:

$$(q-1)\text{-clique extension of } \begin{bmatrix} n \\ 1 \end{bmatrix} \times \begin{bmatrix} m \\ 1 \end{bmatrix}\text{-lattice},$$

where $\begin{bmatrix} n \\ 1 \end{bmatrix} := \frac{q^n-1}{q-1}$.

The 2-clique extension: 

Global structure

Maximal complete subgraphs (**cliques**):

$J_q(n,d)$:

$$\Gamma_1(x) = q\text{-clique extension of } \begin{bmatrix} n-d \\ 1 \end{bmatrix} \times \begin{bmatrix} d \\ 1 \end{bmatrix}\text{-lattice,}$$

$$\swarrow \qquad \searrow$$

$$q\begin{bmatrix} n-d \\ 1 \end{bmatrix} + 1 \quad \text{and} \quad q\begin{bmatrix} d \\ 1 \end{bmatrix} + 1$$

$Bil_q(n \times m)$:

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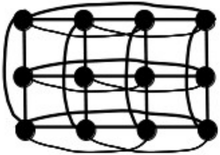

Global structure: $\Gamma \to$ partial linear space

Partial linear space is a set $\mathcal{P}$ of *points* and a set $\mathcal{L}$ of *lines* (subsets of $\mathcal{P}$):

- any line contains at least two points;
- any two points are on at most one line;

Let Γ be $J_q(n,d)$ or $Bil_q(n \times m)$.
Then:

- the set $\mathcal{P}$ of all vertices of Γ,
- the set $\mathcal{L}$ of all maximal cliques of the same size in Γ

form a partial linear space, while Γ is its **point graph**.

A key idea to characterize Γ is to recover the corresponding partial linear space...

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Characterization: partial linear space $\to \Gamma$

.. and then recognize this partial linear space.

Theorem (Ray-Chaudhuri & Sprague, 1976)

Let $(\mathcal{P}, \mathcal{L})$ be a partial linear space s.t. for some $q \geq 2$:

- each line has at least $q^2 + q + 1$ points,
- each point is on more than $q + 1$ lines,
- if $p \in \mathcal{P}$ and $L \in \mathcal{L}$ such that $d(p, L) = 1$, then there are exactly $q + 1$ lines on p meeting L,
- if $p, p' \in \mathcal{P}$ are at distance 2, then there are exactly $q + 1$ lines on p such that $d(p', L) = 1$,
- the point graph Γ of $(\mathcal{P}, \mathcal{L})$ is connected.

Then $(\mathcal{P}, \mathcal{L}, \in) \cong \left(\begin{bmatrix} V \\ d \end{bmatrix}, \begin{bmatrix} V \\ d+1 \end{bmatrix}, \subset\right)$ for a vector space $V = \mathbb{F}_q^n$, and $\Gamma \cong J_q(n, d)$.

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$\iota(\Gamma) \to$ partial linear space

The Bruck/Bose-Laskar/Metsch argument

Suppose that Γ is a graph, regular with valency k and:

- for any pair of adjacent vertices, the number of its common neighbours is exactly λ (relatively large),
- for any pair of non-adjacent vertices, the number of its common neighbours is at most μ (relatively small),
- the valency k is bounded above in terms of λ and μ.

We call a clique of Γ **grand** if it contains $\geq$ some certain number of vertices (depending on k, λ, μ).

Then

- each pair of adjacent vertices is contained in exactly one grand clique ($\to$ **line**),
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In terms of intersection numbers:

$$k = b_0,$$
$$\lambda = a_1,$$
$$\mu = c_2.$$

Two problems

(1) How to find grand cliques?
For $J_q(n,d)$, $Bil_q(n \times m)$, the argument does not work, if (n is close to $2d$) and (n is close to m), respectively.

Recall:

Theorem (Metsch, 1995)

The Grassmann graph $J_q(n,d)$ is characterized by its intersection array if:

- $n \neq 2d$, $n \neq 2d \pm 1$ ($\leftarrow$ the twisted Grassmann),
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Even if we have determined the local structure precisely:

(2) How to recover partial linear space?
In case of equality, $(n = 2d)$ and $(n = m)$, we cannot guarantee the existence of a partial linear space: all maximal cliques have the same size, so we cannot pick out a family of grand cliques (= lines), which cover all the points.

Example (the folded Johnson graph, due to H. Cuypers):
Let Γ be the graph the vertices of which are the pairs of complementary n-subsets of the set $\{1, \ldots, 2n\}$, $n \geq 4$, with two vertices $\{X, Y\}$ and $\{X', Y'\}$ being adjacent iff $|X \cap X'| \in \{1, n-1\}$.

Then $\Gamma_1(\{X, Y\})$ is the $n \times n$-lattice.

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Challenge

If we want to characterize $Bil_q(n \times n)$, we face both problems.

Q-polynomiality and the Terwilliger (subconstituent) algebra theory may help.

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Eigenvalues

The adjacency matrix of a graph Γ:

$$(A)_{x,y} := \begin{cases} 1 \text{ if } x \sim y, \\ 0 \text{ if } x \not\sim y. \end{cases}$$

- The eigenvalues of a graph are the eigenvalues of its adjacency matrix.
- A graph Γ, regular with valency k, has k as an eigenvalue.
- A **non-principal eigenvalue** is an eigenvalue having eigenvector orthogonal to the all-one vector.
- For a graph Γ and a vertex $x \in \Gamma$, a **local eigenvalue at** x is an eigenvalue of $\Gamma_1(x)$, i.e., an eigenvalue of the subgraph induced by the neighbours of x.

Basic theory of DRG

- Let Γ be a DRG with diameter D and on v vertices.
- Define the distance-i matrix A_i of Γ, i.e., $(A_i)_{x,y} = 1$ if $d(x, y) = i$ and 0 otherwise.
- A_1 — the adjacency matrix of Γ.
- One has:

$$A_i A_j = A_j A_i = \sum_{k=0}^{D} p_{ij}^k A_k,$$

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where p_{ij}^k are the intersection numbers.

- This implies that there exist polynomials p_i of degree i, $i = 0, \ldots, D$, such that:

$$A_i = p_i(A_1).$$

Basic theory of DRG

- Let Γ be a DRG with diameter D and on v vertices.
- Define the distance-i matrix A_i of Γ, i.e., $(A_i)_{x,y} = 1$ if $d(x,y) = i$ and 0 otherwise.
- A_1 — the adjacency matrix of Γ.
- One has:

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Basic theory of DRG

- One can show that the matrices $A_0, A_1, \ldots, A_D$ share exactly $D+1$ maximal common eigenspaces, say $W_0, W_1, \ldots, W_D$ so that:

$$\mathbb{C}^v = W_0 \oplus W_1 \oplus \ldots \oplus W_D.$$

- Define $E_j \in \mathbb{C}^{v \times v}$ to be the orthogonal projection onto W_j, so that $E_i E_j = \delta_{ij} E_i$, $\sum_{j=0}^{D} E_j = I$, and

$$A_i = \theta_{i0} E_0 + \theta_{i1} E_1 + \ldots + \theta_{iD} E_D,$$

  where θ_{ij} is an eigenvalue of A_i on W_j.
- $E_0, E_1, \ldots, E_D$ — the **primitive idempotents**.

Basic theory of DRG

- Further, $A_i \circ A_j = \delta_{ij} A_i$, where $\circ$ denotes the entry-wise product.
- This yields that there exist numbers q_{ij}^k such that:

$$E_i \circ E_j = E_j \circ E_i = \sum_{k=0}^{D} q_{ij}^k E_k.$$

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Q-polynomial DRG

- Then there exist polynomials q_j of degree j (with respect to $\circ$-product), $j = 0, \ldots, D$, such that:

$$E_j = q_j(E_1).$$

- Γ is called Q-**polynomial** if there exists an ordering of $E_0, E_1, \ldots, E_D$ such that $E_j = q_j(E_1)$, where q_j is a polynomial of degree j with respect to entry-wise product of matrices.

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- *All known* families of (primitive) DRG with unbounded diameter are Q-polynomial.

The Terwilliger polynomial

- Let Γ be a Q-polynomial DRG with diameter $D \geq 3$.
- Terwilliger (early 1990's): There exists a polynomial p_T of degree 4 such that, for any vertex $x \in \Gamma$, and any non-principal eigenvalue η of $\Gamma_1(x)$ we have $p_T(\eta) \geq 0$.
- p_T only depends on the intersection numbers and the Q-polynomial ordering of $E_0, E_1, \ldots, E_D$.
- We call p_T the **Terwilliger polynomial**.

The Terwilliger polynomial was described in '*Lecture Note on Terwilliger algebras*', the lectures given by P. Terwilliger in Japan and recorded by H. Suzuki.

The Terwilliger polynomial

$$p_T(\eta) := p_1(\eta)p_2(\eta) - p_3(\eta)^2,$$

where

$$p_1(\eta) = -\eta^2 + (a_1 - c_2)\eta + (k - c_2),$$

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$$p_2(\eta) = p^1_{23}\Big(\tau_0\eta^2 + (\tau_1 - \tau_2)\eta + (1 - a_1\tau_0 - \tau_2)\Big)$$

$$\tau_0 = \frac{1}{b_1} \times \frac{(\theta^*_2 - \theta^*_1)(\theta^*_0 + \theta^*_1 - \theta^*_2 - \theta^*_3)}{(\theta^*_0 - \theta^*_2)(\theta^*_3 - \theta^*_2)},$$

$$\tau_1 = \frac{\theta^*_2 - \theta^*_1}{\theta^*_3 - \theta^*_2}\Big(\frac{(\theta^*_1 - \theta^*_3)}{(\theta^*_0 - \theta^*_2)} - \frac{a_1 - 1}{b_1} - \frac{(a_1 - 1)(\theta^*_1 - \theta^*_3)}{b_1(\theta^*_0 - \theta^*_2)}\Big),$$

$$\tau_2 = \frac{\theta^*_2 - \theta^*_1}{\theta^*_3 - \theta^*_2}\Big(\frac{(\theta^*_1 - \theta^*_3)}{(\theta^*_0 - \theta^*_2)} - \frac{a_1 + 1 - c_2}{b_1} - \frac{(a_1 + 1)(\theta^*_1 - \theta^*_3)}{b_1(\theta^*_0 - \theta^*_2)} + \\ + \frac{(\theta^*_1 - \theta^*_2)^2 - (\theta^*_0 - \theta^*_1)(\theta^*_2 - \theta^*_3)}{(\theta^*_0 - \theta^*_1)(\theta^*_0 - \theta^*_2)} - \frac{1}{b_1} \times \frac{(\theta^*_0 - \theta^*_1)(\theta^*_0 + \theta^*_1 - \theta^*_2 - \theta^*_3)}{(\theta^*_0 - \theta^*_2)(\theta^*_1 - \theta^*_2)}\Big),$$

where

$$E_1 = \theta^*_0 A_0 + \theta^*_1 A_1 + \ldots + \theta^*_D A_D.$$

$Bil_q(n \times n)$

Let Γ be a DRG with the same intersection array as $Bil_q(n \times n)$.

- The Terwilliger polynomial p_T has the three distinct roots:

$$-q-1, -1, \text{ and } q^n - q - 1 \text{ (of multiplicity 2)},$$

while the leading coefficient of p_T is negative.

- Hence a local non-principal eigenvalue η at any vertex $x \in \Gamma$ satisfies:

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For $q = 2$, a local non-principal eigenvalue η at any vertex $x \in \Gamma$ satisfies:

$$\eta \in [-3, -1] \text{ or } \eta = 2^n - 3.$$

Claim

$\Gamma_1(x)$ has integral eigenvalues, i.e., -3, -2, -1, or $2^n - 3$.

Proof:

1. The eigenvalues of a graph are algebraic integers, and their product is an integer.

2. Let $\eta_1, \ldots, \eta_s$ be all *irrational* eigenvalues of $\Gamma_1(x)$. Then $\eta_i \in (-3, -1)$ and $\Pi_{i=1}^s \eta_i$ is an integer, so

$$\Pi_{i=1}^s (\eta_i + 2)$$

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$$\eta_i \in (-3, -1) \Rightarrow |\eta_i + 2| < 1 \Rightarrow \Pi_{i=1}^s (\eta_i + 2) = 0.$$

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Recall a basic fact:

Let $\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_s^{m_s}$ be the spectrum of a regular (with valency k) graph on v vertices, and A be its adjacency matrix.
Then:

$$\sum_{i=0}^{s} m_i = v,$$

$$tr(A) = \sum_{i=0}^{s} m_i \theta_i = 0,$$

$$tr(A^2) = \sum_{i=0}^{s} m_i \theta_i^2 = vk.$$

We may put $\theta_0 = k$ and, moreover, $m_0 = 1$ if the graph is connected.

$Bil_2(n \times n)$

Claim

$\Gamma_1(x)$ has spectrum $2(2^n-2)^1,\ (2^n-3)^{2(2^n-2)},\ (-2)^{(2^n-1)^2}$.

Proof:

Apply the previous fact to our case:

$$b_0 = v = (2^n-1)^2, \quad \theta_0 = k = a_1 = 2(2^n-2),$$

$$\theta_1 = 2^n-3, \quad \theta_2 = -1, \quad \theta_3 = -2, \quad \theta_4 = -3,$$

$$m_1 = ?, \quad m_2 = ?, \quad m_3 = ?, \quad m_4 = ?,$$

and $m_0 = 1$ (as $\Gamma_1(x)$ is connected).

The system of (three) linear equations with respect to (four) unknowns $m_1, \ldots, m_4$ has the only non-negative integral solution:

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$\Gamma_1(x)$ is a regular graph with exactly 3 distinct eigenvalues

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$\Gamma_1(x)$ is DRG with diameter 2 (*strongly regular graph*)

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$\Gamma_1(x)$ has the parameters of the lattice graph $H(2, 2^n - 1)$

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So, Γ has the same local graphs as $Bil_2(n \times n)$.

In other words, for a any vertex $\mathbf{x}$ of $\mathcal{B} := Bil_2(n \times n)$ and any vertex $x \in \Gamma$, we have an isomorphism:

$$\varphi : \ \{\mathbf{x}\} \cup \mathcal{B}_1(\mathbf{x}) \to \{x\} \cup \Gamma_1(x).$$

Then we are able to show this isomorphism may be extended to:

$$\varphi' : \ \{\mathbf{x}\} \cup \mathcal{B}_1(\mathbf{x}) \cup \mathcal{B}_2(\mathbf{x}) \to \{x\} \cup \Gamma_1(x) \cup \Gamma_2(x),$$

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$Bil_2(n \times n)$

Then the theorem* by Munemasa and Shpectorov shows that $\Gamma \cong \mathcal{B}$.

(In order to use this theorem, we need to check some additional assumptions, which indeed hold in our case.)

* A. Munemasa, S.V. Shpectorov, A local characterization of the graphs of alternating forms // Finite geometry and combinatorics, 1993. P. 289-302.

Problems: $q = 3$

To characterize the next case, $Bil_3(n \times n)$, one has to solve the following problems:

- Does $\Gamma_1(x)$ have only integer eigenvalues?
  In case of $Bil_3(n \times n)$, $p_T(\eta)$ shows that the non-principal local eigenvalues satisfy
  $$\eta \in [-4, -1] \text{ or } \eta = 3^n - 4.$$
- Does $\Gamma_1(x)$ have spectrum
  $(2 \cdot 3^n - 5)^1, (3^n - 4)^{\frac{3^n-3}{2}}, (-1)^{\frac{(3^n-1)^2}{4}}, (-3)^{\frac{(3^n-3)^2}{4}}$?
- Is $\Gamma_1(x)$ the 2-clique extension of $\frac{3^n-1}{2} \times \frac{3^n-1}{2}$?
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  In case of $Bil_3(n \times n)$, $p_T(\eta)$ shows that the non-principal local eigenvalues satisfy
  $$\eta \in [-4, -1] \text{ or } \eta = 3^n - 4.$$
- Does $\Gamma_1(x)$ have spectrum
  $(2 \cdot 3^n - 5)^1, (3^n - 4)^{\frac{3^n-3}{2}}, (-1)^{\frac{(3^n-1)^2}{4}}, (-3)^{\frac{(3^n-3)^2}{4}}$?
- Is $\Gamma_1(x)$ the 2-clique extension of $\frac{3^n-1}{2} \times \frac{3^n-1}{2}$?
- Do we have an extendable isomorphism?

Problems: doubtful intersection arrays

The intersection array

$$\{7(M-1), 6(M-2), 4(M-4); 1, 6, 28\}$$

is feasible for all integer $M \geq 6$.

The only known graphs with this array: $Bil_2(3 \times m)$, where $M = 2^m$.

- $M = 6$: ruled out by Jurisic & Vidali (2012) by counting some triple intersection numbers,
- $M = 7$: the first open case,
- $M = 8$: $Bil_2(3, 3)$,
- ...
- By the result of Metsch, we may assume $M < 133$.

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Problems

Applying the Terwilliger polynomial:

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$$\{35, 24, 8; 1, 6, 28\}$$

The spectrum of $\Gamma_1(x)$ would be $10^1, 3^{13}, -1^7, -3^{14}$.

But it is unfeasible, as the number of triangles through a vertex:

$$\Delta = \frac{1}{2 \cdot 35}(10^3 + 13 \cdot 3^3 + 7 \cdot (-1)^3 + 14 \cdot (-3)^3)$$

is not integer!

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- $M = 7$ ($\exists$?):
$$\{42, 30, 12; 1, 6, 28\}$$
The spectrum of $\Gamma_1(x)$ would be $11^1, 4^{12}, -1^{14}, -3^{15}$.
(Un)fortunately, this spectrum is feasible: the number of closed walks
$$\frac{1}{42}(11^l + 12 \cdot 4^l + 14 \cdot (-1)^l + 15 \cdot (-3)^l)$$
is integer for $l = 2, 4, 6, \ldots$, and even integer for $l = 3, 5, \ldots$.

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More results

In this project we also showed that the following Q-polynomial DRGs are characterized by their intersection arrays:

- the folded Johnson graphs $\tilde{J}(2m, m)$, $m \geq 6$.
- the halved folded cubes $\frac{1}{2}\tilde{H}(2m, 2)$, $m \geq 6$.
- the Grassmann graphs $J_2(2D, D)$ with odd D.